Three Realisms

and

The Idea of Sheaves

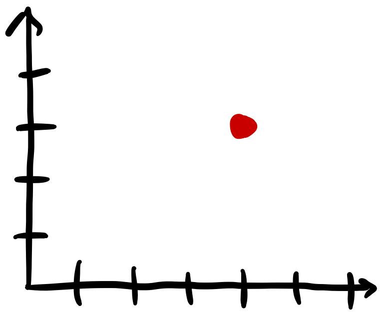
David Jaz Myers
JHU

Three Realisms: (and their mathematical tools)

1. Fixed Realism (Sets)
   - One model
   - What's real is what's true in that model.
2. Covariant Realism (Group actions)
   - Many equivalent models
   - What's real is how things change as you change your model
3. Local Realism (Sheaves)
   - Many inequivalent models
   - What's real is how you handle disagreement (Cohomology)

Fixed Realism

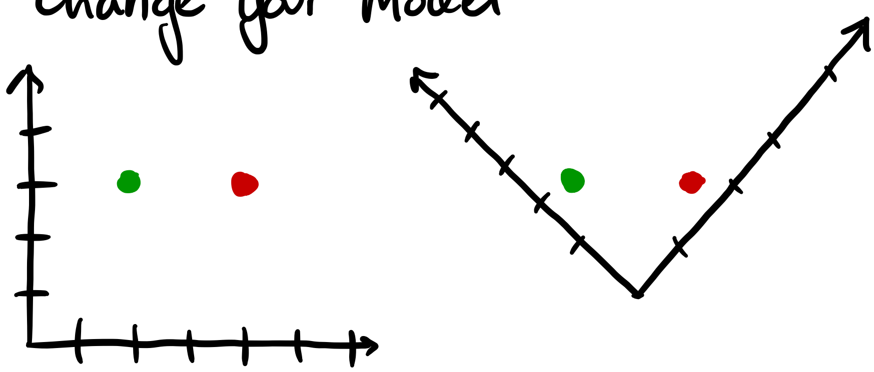
- One model
- What's real is what's true in that model.



The dot is at (4, 3)

Covariant Realism

- Many equivalent models
- What's real is how things change as you change your model



The dots are two units apart.

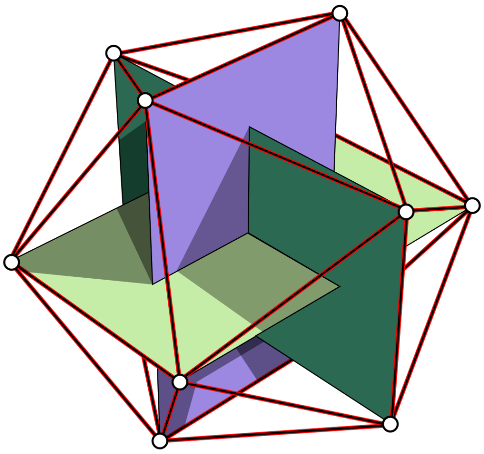
If we have different coordinates, we need to coordinate! (D. Spivak)

We need an <u>equivalence</u> $e : C_1 \cong C_2$

These form a <u>group</u>. (of symmetries)

Covariant Realism: Symmetry

The facts depend on what kind of equivalence:
What is our group of symmetries?

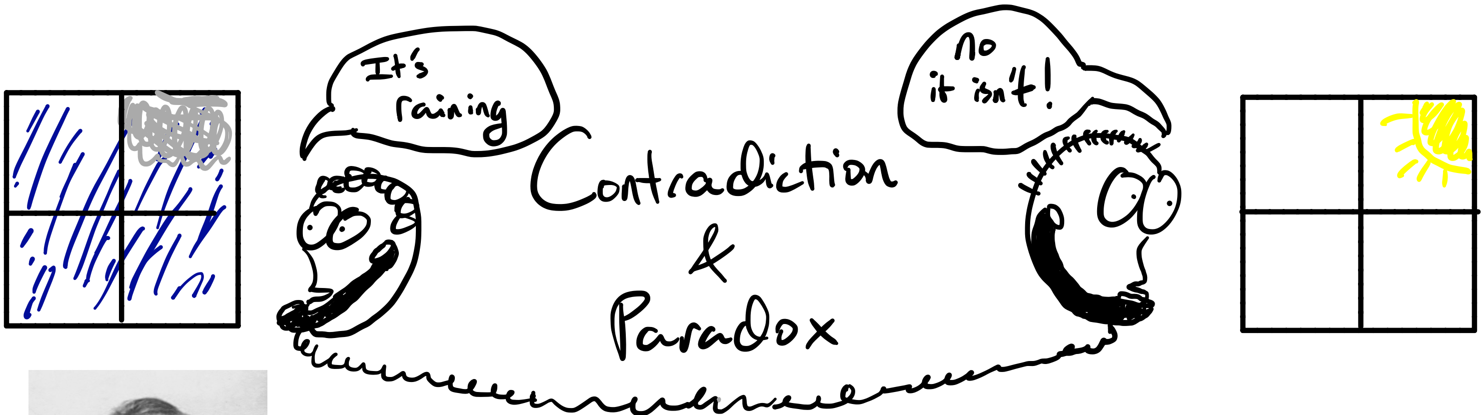


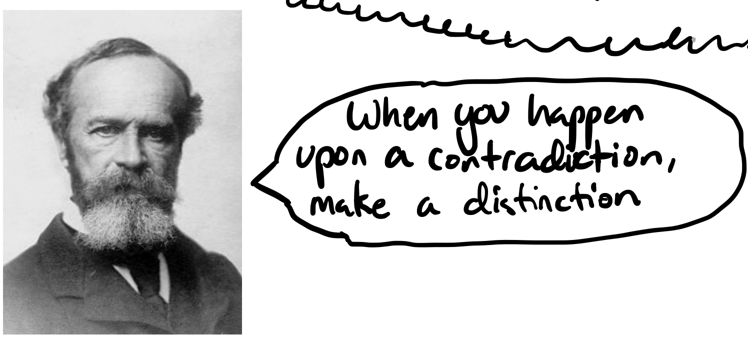
Klein: your group determines your geometry

Noether: your group determines your conserved quantities

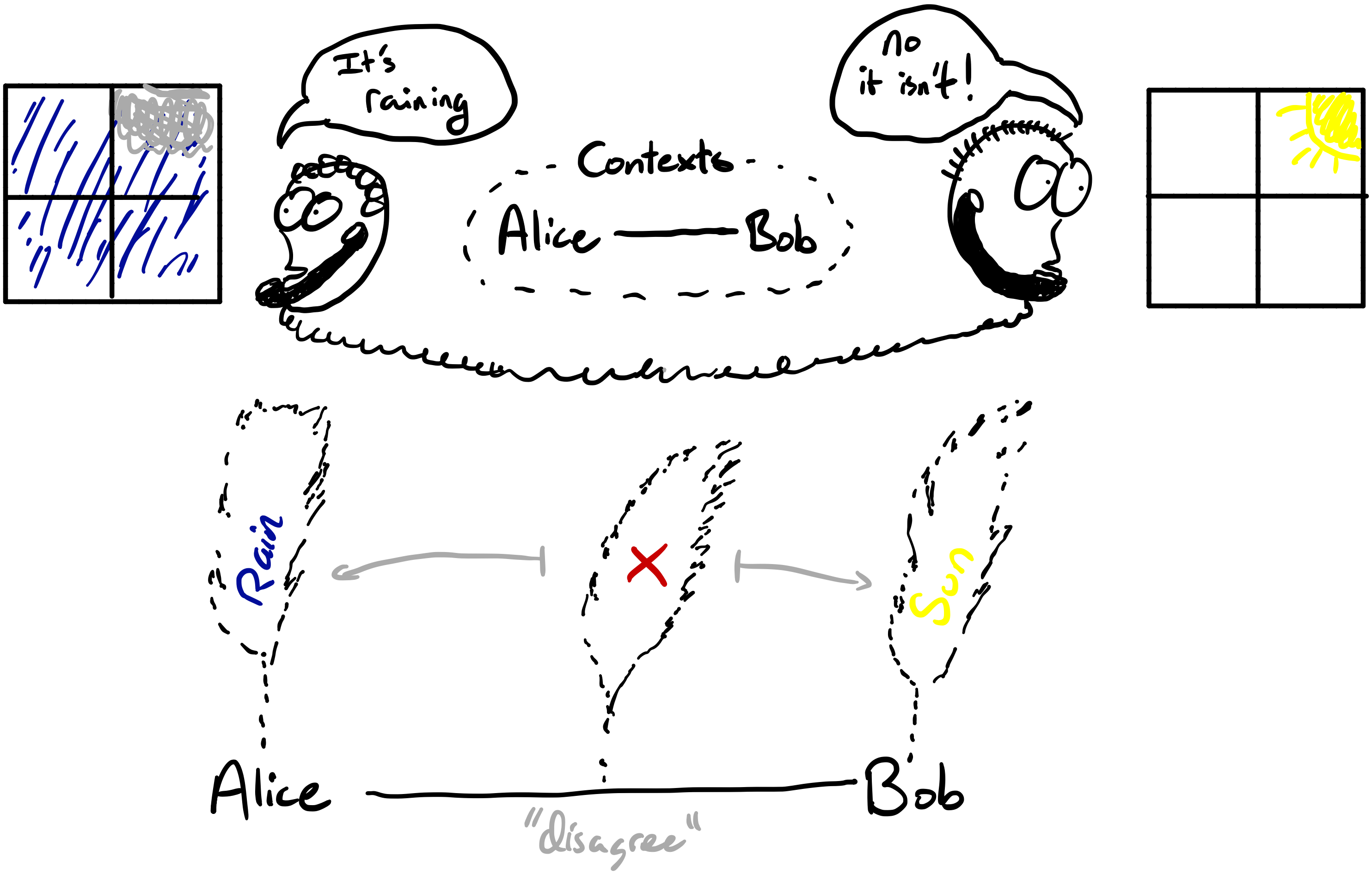
Local Realism

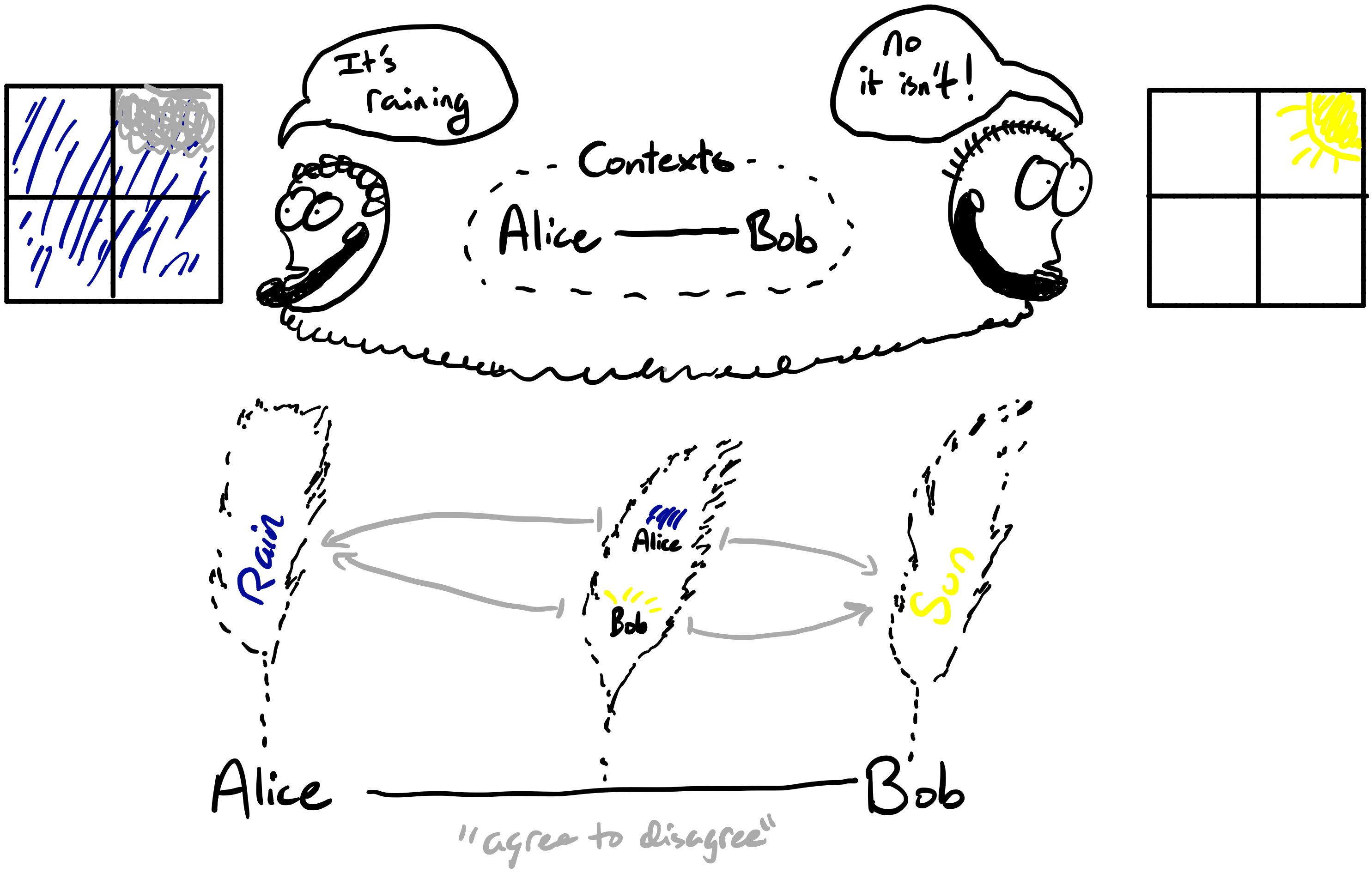
- Many inequivalent models
- What's real is how you handle disagreement

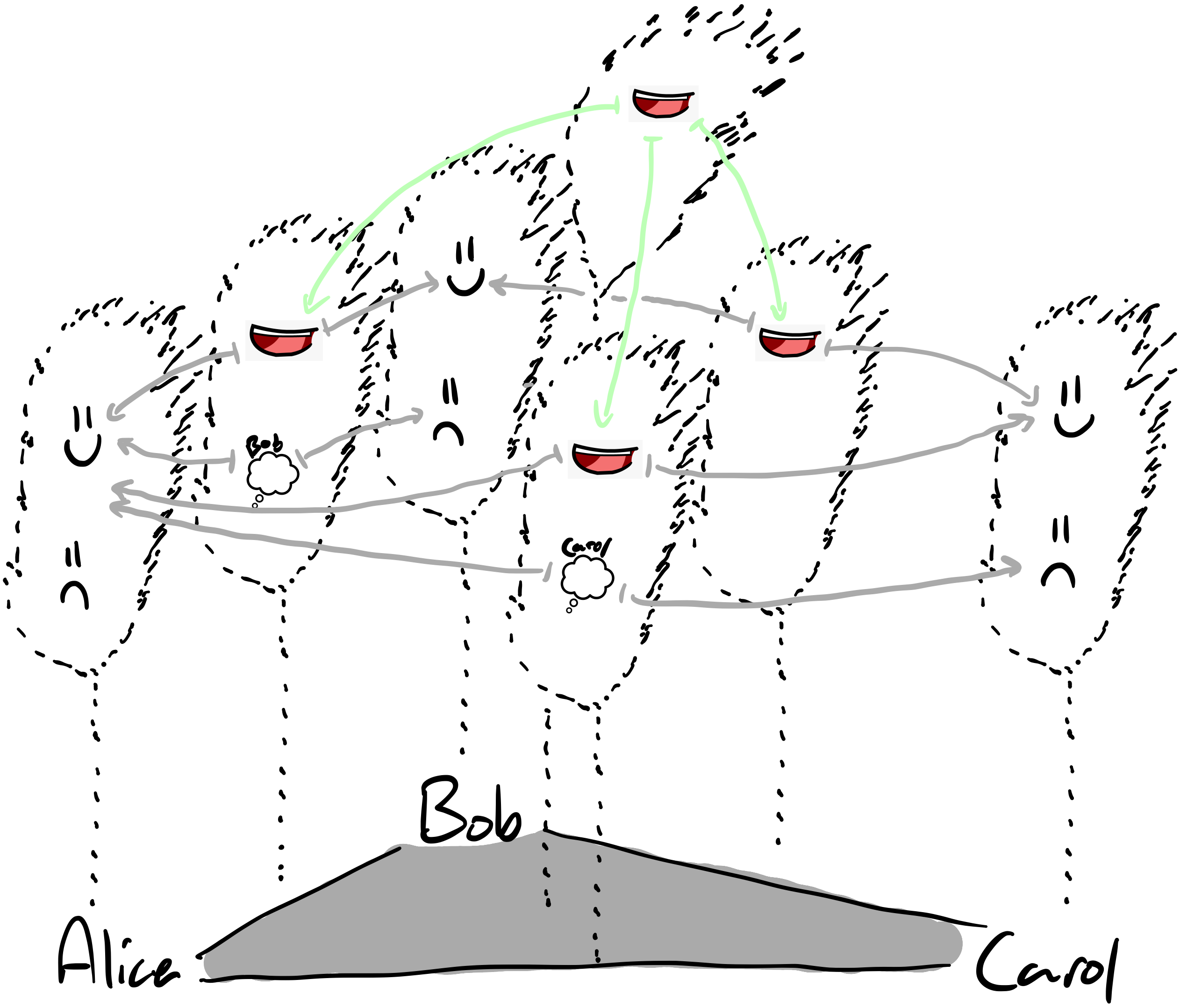


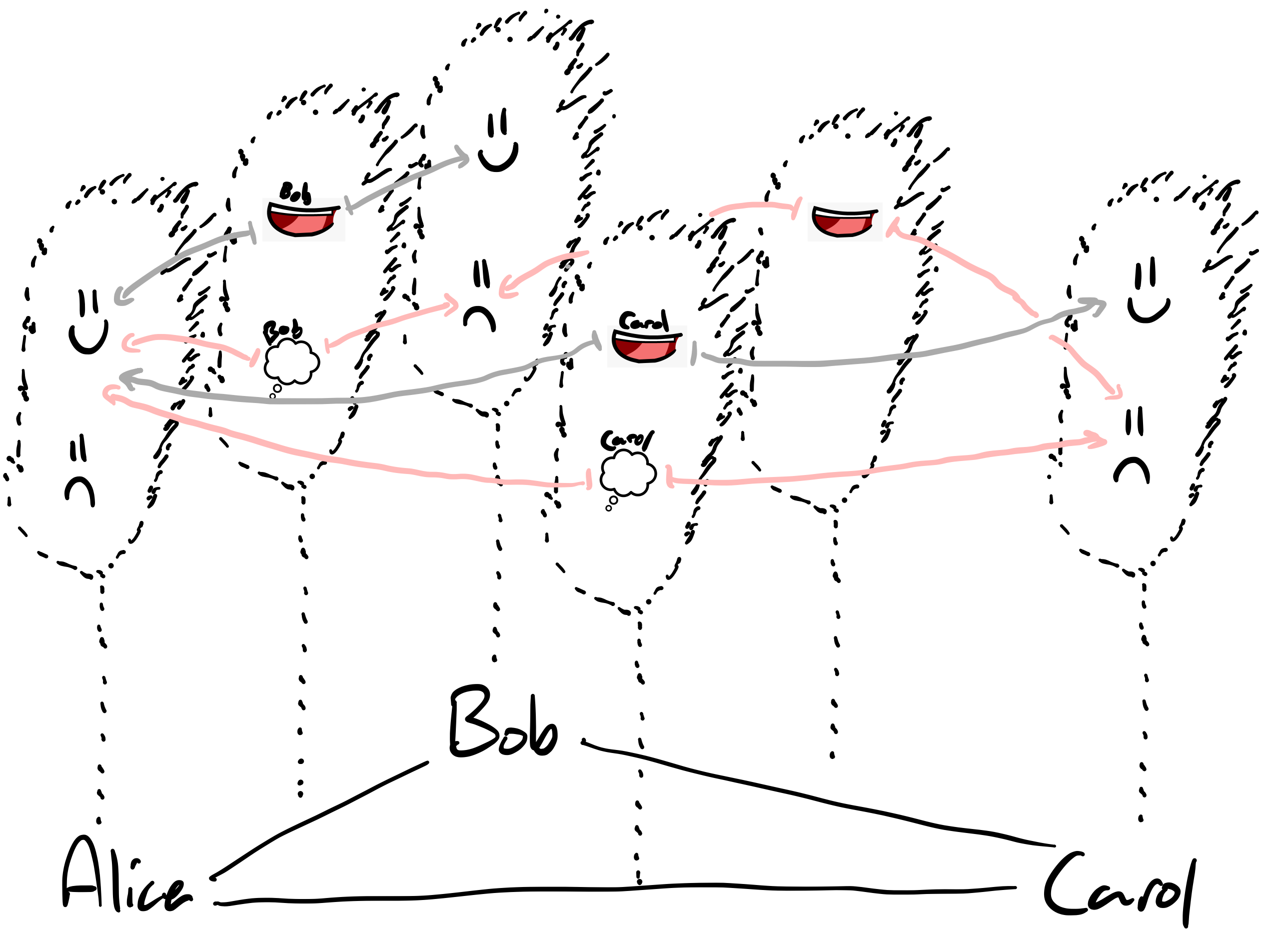


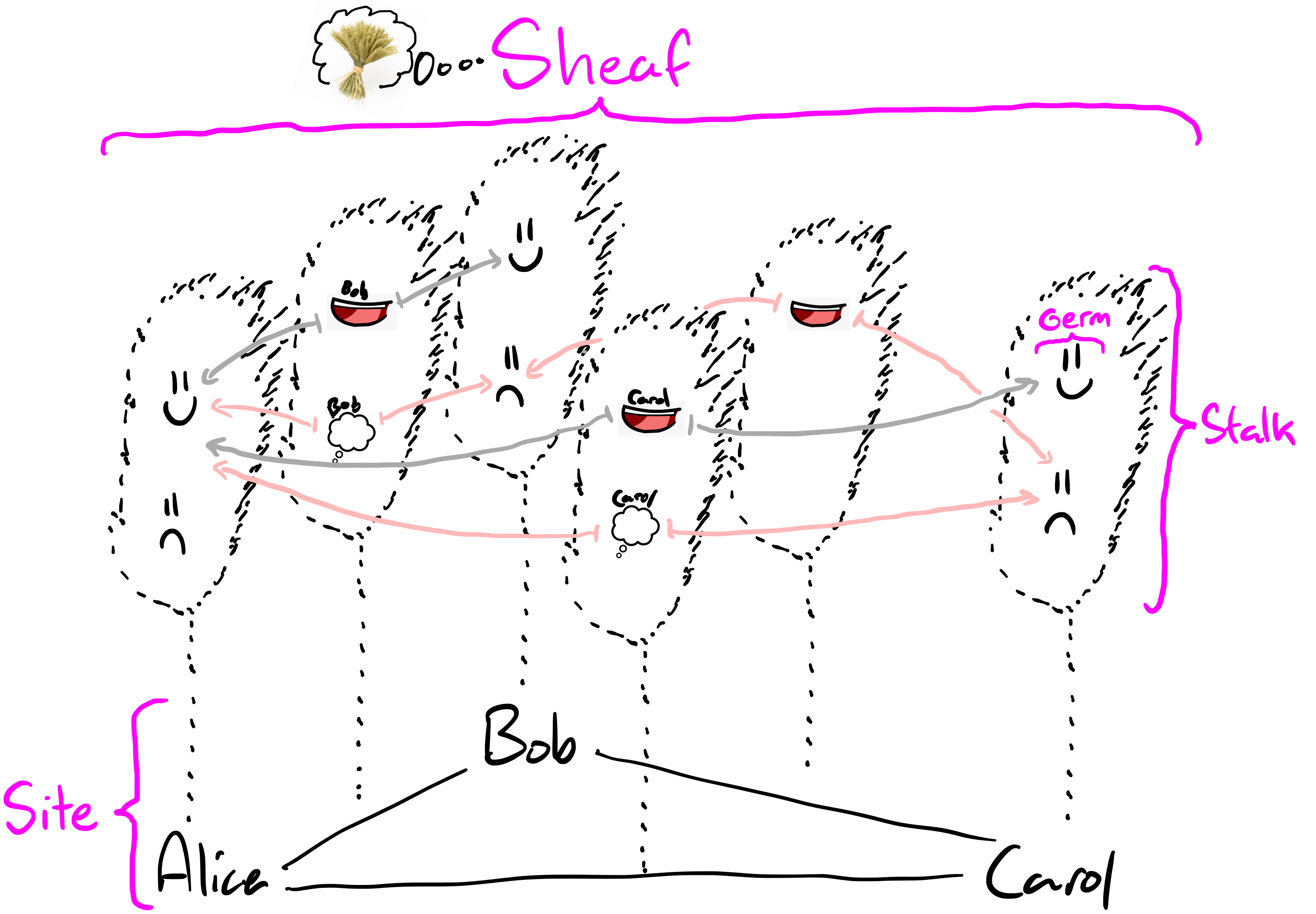
⇝ distinct contexts

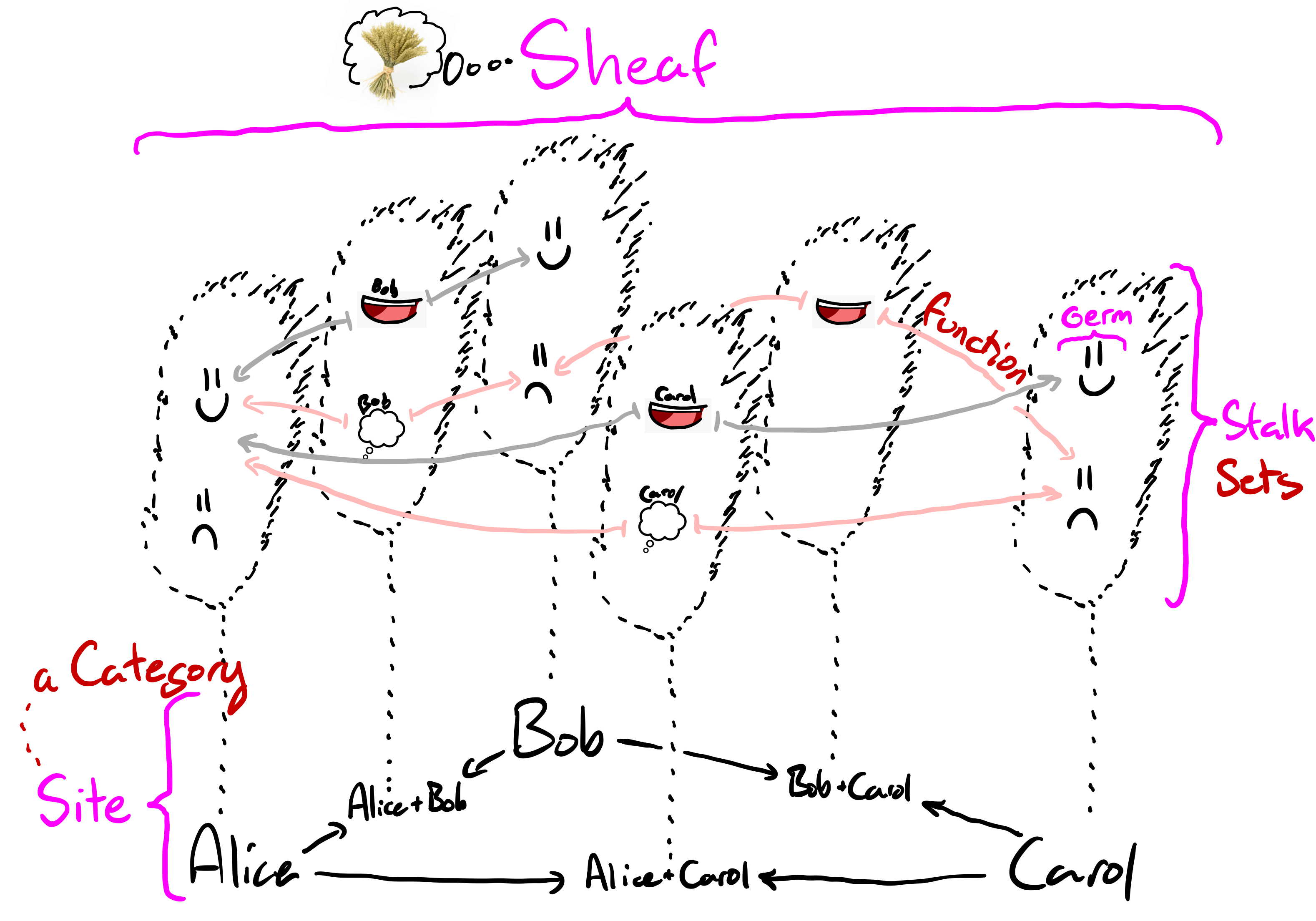

It's raining
No it isn't!
Contexts
Alice
Bob
Rain
Sun
Alice
Bob
"disagree"


It's raining
No it isn't!
Contexts
Alice
Bob
Rain
Alice
Bob
Sun
Alice
Bob
"agree to disagree"


Bob
Carol
Bob
Alice
Carol


Bob
Bob
Carol
Carol
Bob
Alice
Carol


Sheaf
Bob
Bob
Carol
Carol
Germ
Stalk
Site
Bob
Alice
Carol


Sheaf
Bob
Bob
function
Carol
Carol
Germ
Stalk
Sets
a Category
Site
Bob
Alice+Bob
Bob+Carol
Alice
Alice+Carol
Carol

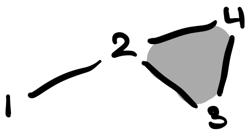
Sheaves

Def: A (pre)sheaf on a site $\mathcal{C}$ (category "of contexts")
is a contravariant functor $M : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

- A model $M(C)$ for each context $C \in \mathcal{C}$
- For an inclusion of contexts $i : C \to C'$, a specialization of models $M(C') \longrightarrow M(C)$
- if
$$\begin{tikzcd} & C' \arrow[dr] & \\ C \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "="] & & C'' \end{tikzcd}$$
, then
$$\begin{tikzcd} & M(C') \arrow[dl] & \\ M(C) \arrow[rr, phantom, shift left=3, "="] & & M(C'') \arrow[ul] \arrow[ll] \end{tikzcd}$$
- Gluing: important but not going to cover it.

Sheaves on a simplicial complex.

A simplicial complex with set of points P is a set of <u>cells</u> $C \subseteq 2^P$ where every subset of a cell is a cell.

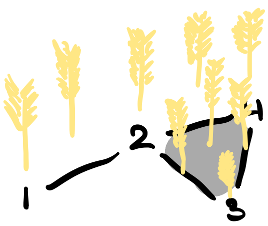
$C =$  is

$$\begin{tikzcd} & & & \{2,3,4\} & & \\ \{1,2\} & \{2,3\} \arrow[urr] & & \{2,4\} \arrow[u] & & \{3,4\} \arrow[ull] \\ \{1\} \arrow[u] & \{2\} \arrow[ul] \arrow[u] \arrow[urr] & & \{3\} \arrow[ull] \arrow[urr] & & \{4\} \arrow[ull] \arrow[u] \end{tikzcd}$$

C is a category!

Warning!
↳ AKA "Cosheaf"

A sheaf on C is a (pre)sheaf on this category

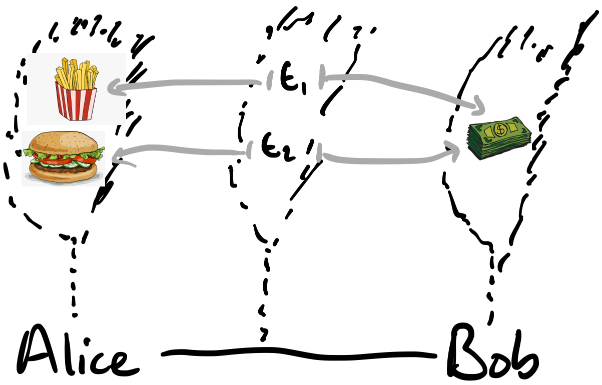


$$\begin{tikzcd} & & & M(2,3,4) \arrow[dll] \arrow[d] \arrow[drr] & & \\ M(1,2) \arrow[d] \arrow[dr] & M(2,3) \arrow[d] \arrow[drr] & & M(2,4) \arrow[dll] \arrow[drr] & & M(3,4) \arrow[dll] \arrow[d] \\ M(1) & M(2) & & M(3) & & M(4) \end{tikzcd}$$

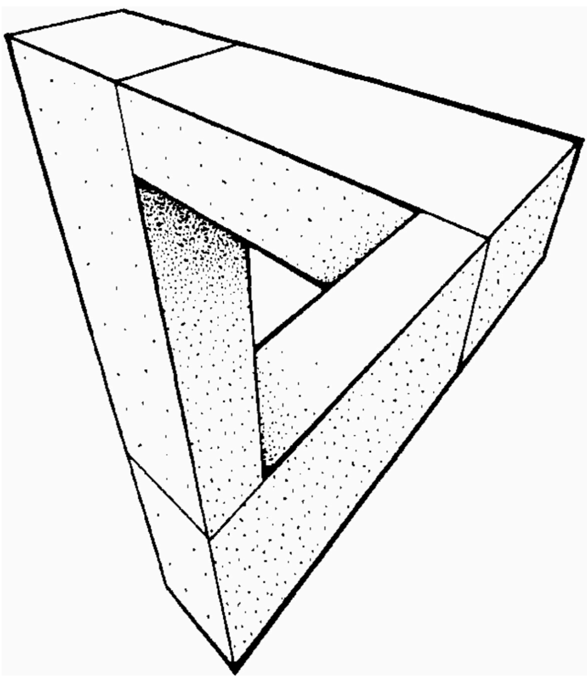
Sheaves on a graph: Market Models

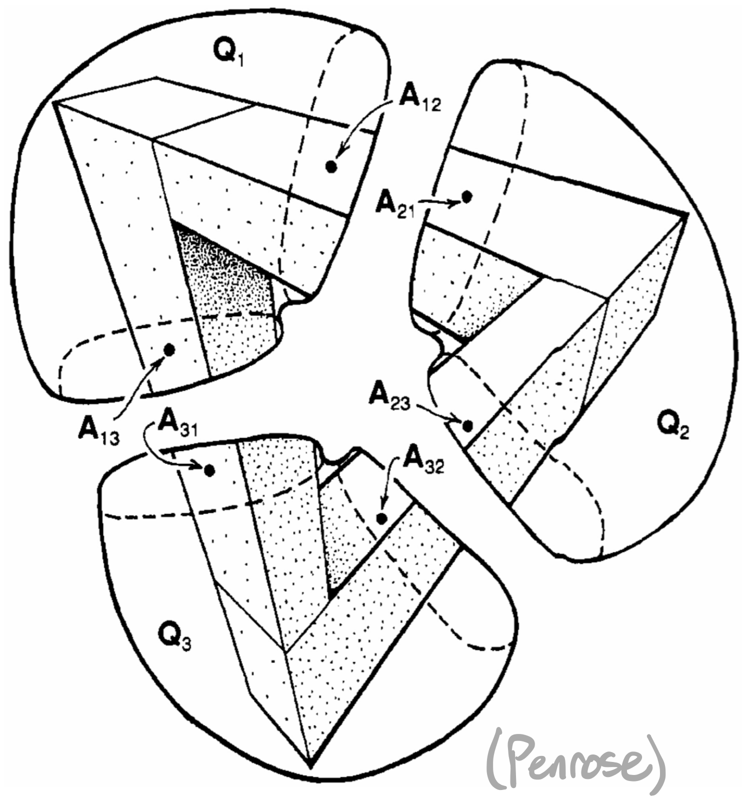
A *market model* has:

- (Site) *Agents*, connected by *Channels* in a graph.
- (Sheaf) Each agent has a set of *goods* or *baskets* it can trade, and each channel has a set of *transactions* that can occur over it.



Cohomology: Consensus and disagreement



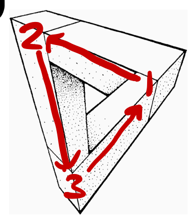


(Penrose)

Cohomology: Consensus and disagreement

Cohomology lets us measure disagreement with linear algebra

For a sheaf $M : \mathcal{C}^{op} \to \mathrm{Set}$ on a graph $\mathcal{C}$, oriented



Cochains

$C^0(M) := \text{Functions } \coprod_{\substack{\text{points}\\ \text{in } \mathcal{C}}} M(p) \longrightarrow \mathbb{R}$ e.g. Distances

$C^1(M) := \text{Functions } \coprod_{\substack{\text{edges}\\ \text{in } \mathcal{C}}} M(e) \longrightarrow \mathbb{R}$ e.g. relative distances

φ:
$\overline{12} \longmapsto 1$
$\overline{23} \longmapsto 1$
$\overline{31} \longmapsto 1$

differential

$C^0(M) \xrightarrow{d} C^1(M) \qquad df(e) := f(e_1) - f(e_0)$

$\varphi \overset{?}{=} df$?

Cohomology

$H^0(M) = \ker d = \{ f \in C^0(M) \mid df = 0 \}$

$H^1(M) = \operatorname{Coker} d = \{ \varphi \in C^1(M) \} / \varphi \sim \varphi + df$

df:
$\overline{12} \longmapsto f(2) - f(1)$
$\overline{23} \longmapsto f(3) - f(2)$
$\overline{31} \longmapsto f(1) - f(3)$

so $df(\overline{12}) + df(\overline{23}) = -df(\overline{31})$
but not true for φ!

Cohomology: The possibility of arbitrage

In a market model M

$C^0(M) = \{ \coprod_{\text{agents}} \{\text{baskets}\} \longrightarrow \mathbb{R}^+ \}$ are pricing functions

$C^1(M) = \{ \coprod_{\text{Channels}} \{\text{transactions}\} \longrightarrow \mathbb{R}^+ \}$ are exchange rates

Thm: Let $r \in C^1(M)$ be an exchange rate. Then

Arbitrage is possible — A loop of transactions (t_i) with $\prod_i r_i > 1$

iff

There is no global price — A $p \in C^0(M)$ with $dp = r$

$dp(t) = p(t_1)/p(t_0)$

Cohomology: The dynamics of disagreement.

Disagreement leads to change
- Arbitrage opportunities cause changes in local prices
- Water flows downhill
- Disagreements can cause changes in opinion [1]

But these are all changes in a sheaf.

What about changes of sheaf? — new goods new transactions

What about changes of site? — new markets

Other things sheaves do:

Thanks

- Poly-Systems are presheaves [2]
- Variable Sharing Systems are sheaves [3]
- Databases are presheaves [4]
- Sheaves model dependent type theories
- Sensor Integration [5,6]
- All sorts of "spaces" (e.g. manifolds) are sheaves
- ⋮
- Yoneda Embedding: anything is a sheaf if you want it to be.
- Not to mention enriched presheaves... oh my!

Further Reading

[1] OPINION DYNAMICS ON DISCOURSE SHEAVES*
JAKOB HANSEN† AND ROBERT GHRIST‡

[2] Poly: An abundant categorical setting for mode-dependent dynamics
David I. Spivak

[3] Sheaf Semantics for Concurrent Interacting Objects
Joseph A. Goguen*

[4] FUNCTORIAL DATA MIGRATION
DAVID I. SPIVAK

[5] Sheaves are the canonical data structure for sensor integration
Michael Robinson

[6] Target enumeration via integration over planar sensor networks
Yuliy Baryshnikov Robert Ghrist

Variable Quantities and Variable Structures in Topoi
F. WILLIAM LAWVERE