A Short Introduction to Categorical Logic

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Why logical pluralism?

Most mathematicians and philosophers know about **first-order logic**:

$$\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \forall, \exists, =$$

Shouldn't that be good enough for everyone?

No, because it's *too general*:

- Simple kinds of theories are needlessly complicated and their structure lost, e.g., the equational character of algebraic theories
- The richer the logical system, the fewer models its theories can have

Yet it's also *not general enough*:

- Does not easily accomodate unconventional semantics
- E.g., nondeterminism or resource-boundedness

Thus, we should take a **pluralistic** view of logic.

Structuralism for logic

A hallmark of modern mathematics is its **structuralist** approach:

- rather than studying specific objects and their properties (e.g., $\mathbb{N}$ and $\mathbb{R}$),
- study structures and their relations (e.g., groups and rings)
- with algebra playing a central role

Categorical logic is a way of being structuralist about logic itself [Awo96].

It is *anti-reductionist* and arguably *anti-foundationalist*:

- rather than seeing mathematics as something built on top of logic,
- logic becomes part of mathematics itself
- studied using algebra and, in particular, category theory

Dictionary between category theory and logic

Category theory	Logic
Category $\mathcal{C}$	Theory
Functor $\mathcal{C} \longrightarrow \mathcal{S}$	Model
Natural transformation	Model homomorphism

To be more precise, in this dictionary:

- Categories usually have extra structure
- Functors and natural transformations preserve this structure

Different ways of choosing this extra structure give different logical systems.

Note. While the study of classical abstract algebra is 1-categorical, the study of cateogrical logic is properly 2-categorical.

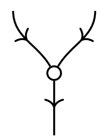
Algebraic theories

Categorical logic began with Lawvere's study of algebraic theories [Law63, Cro93].

Definition. *A* Lawvere theory *is a small cartesian monoidal category whose objects are freely generated by one object.*

Lawvere theories represent single-sorted algebraic theories in a syntax-invariant way.

Example. The *theory of monoids* Th(Mon) is generated by the morphisms



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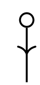
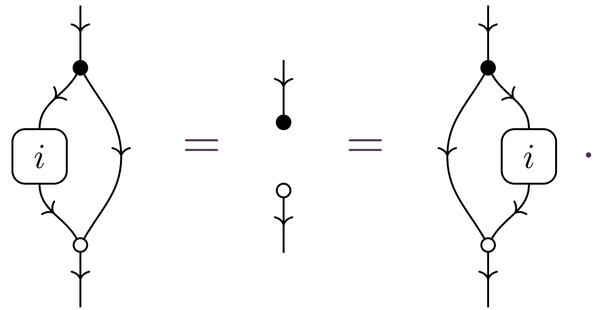


subject to the equations

Algebraic theories

Example. The *theory of groups* Th(Grp) extends Th(Mon) with another morphism $i\colon x \longrightarrow x$ subject to the equations



Definition. *A* model *of a Lawvere theory* $\mathcal{C}$ *is a cartesian monoidal functor* $\mathcal{C} \longrightarrow$ Set*.*

E.g, a model of Th(Grp) is a group. Moreover, the monoidal natural transformations between models are group homomorphisms.

Invariance of Lawvere theories

We gave the standard presentation of the theory of groups, but it has many different axiomatizations, such as:

Example. A *group* is a set G with a binary operation $(g,h) \mapsto g/h$ and a constant e such that $g/g = e$, $g/e = g$, and $(g/k)/(h/k) = g/h$ for all $g,h,k \in G$.

Correspondingly, present a Lawvere theory $\mathsf{Th}(\mathsf{Grp})'$ with morphisms $\delta\colon x \otimes x \to x$ and $\eta\colon I \to x$ subject to three equations.

The two Lawvere theories are not equal but they are isomorphic:

$$\mathsf{Th}(\mathsf{Grp}) \cong \mathsf{Th}(\mathsf{Grp})'.$$

In general, theories in categorical logic are not *syntactical* objects but *algebraic* ones, hence they are **invariant** to differences of presentation.

Functorial semantics

So far we have considered semantics in $\mathcal{S} = \mathsf{Set}$, but the semantics category $\mathcal{S}$ can be any category with the required structure.

Definition. *A* model *of a Lawvere theory $\mathcal{C}$ in a cartesian category $\mathcal{S}$ is a cartesian functor $\mathcal{C} \to \mathcal{S}$.*

This powerful notion of **functorial semantics** is distinctive of categorical logic.

Example. A *group object* in a cartesian category $\mathcal{S}$ is a model of $\mathsf{Th}(\mathsf{Grp})$ in $\mathcal{S}$.

A group object in...	is a...
Set	group
Top	topological group
Man	Lie group
G-Set	semidirect product $(-) \rtimes G$

Monoidal theories

The weaker the logicial system, the more categories $\mathcal{S}$ can serve as its semantics.

Thus, it is useful to consider logical systems weaker than Lawvere theories.

Doctrine	Single-sorted	Typical theories
category	—	discrete dynamical systems
monoidal category	PRO	(co)monoids
symmetric monoidal category	PROP	commutative (co)monoids
cartesian category	Lawvere theory	groups, rings

Example. A *monoid object* in a monoidal category $\mathcal{S}$ is a model of $\mathsf{Th}(\mathsf{Mon})$ in $\mathcal{S}$, where $\mathsf{Th}(\mathsf{Mon})$ is now regarded as PRO.

Infamously, a *monad* on a category $\mathcal{C}$ is a monoid object in $(\mathrm{End}_{\mathcal{C}}, \circ, 1_{\mathcal{C}})$.

On the other hand, the theories of commutative monoids and of groups cannot be interpreted in $(\mathrm{End}_{\mathcal{C}}, \circ, 1_{\mathcal{C}})$.

A family tree of categorical logic

category

↓

monoidal category
(PRO)

↓

symmetric monoidal category
(PROP)

↓

cartesian category
(algebraic theory / Lawvere theory)

↓

regular category
(regular logic: $\exists, \wedge, \top$)

↓

coherent category
(coherent logic: $\exists, \wedge, \vee, \top, \bot$)

↓

elementary topos
(first-order logic)

cartesian category → cartesian closed category
(lambda calculus + product types)

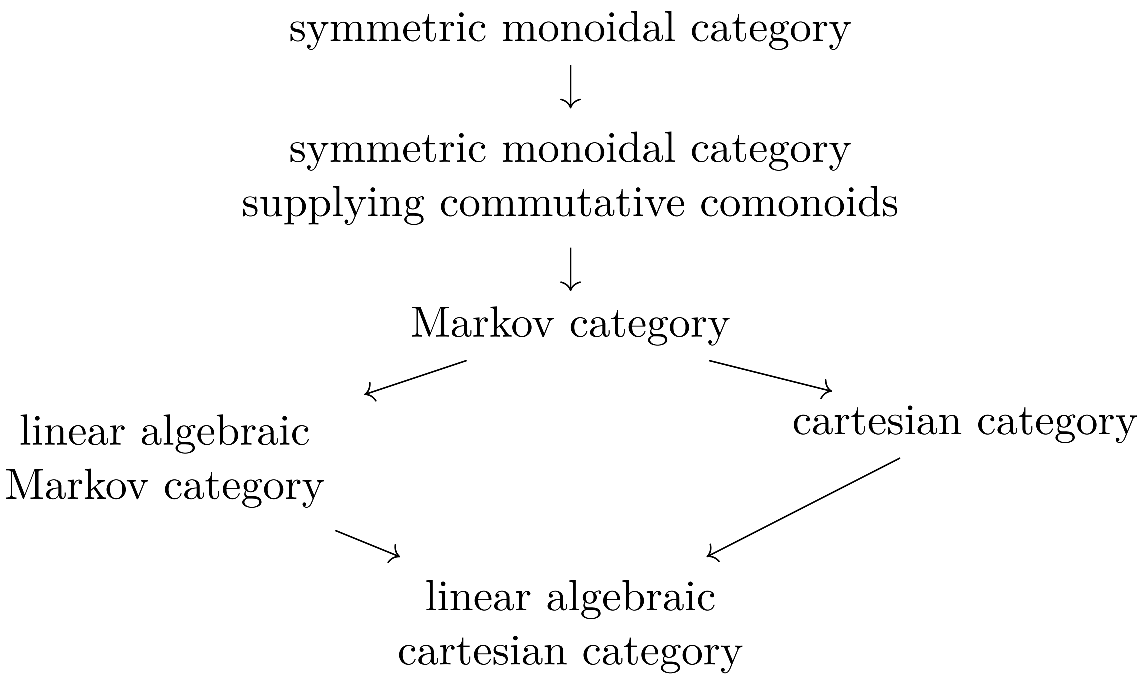
↓

bicartesian closed category
(lambda calculus + product and sum types)

→ elementary topos

[Pat20, Figure 1.1]

Another branch of the family tree



[Pat20, Figure 1.2]

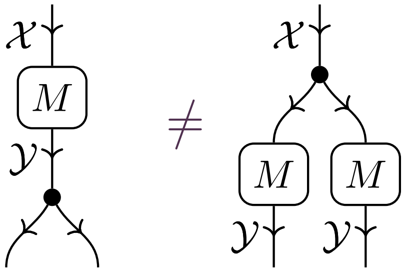
Markov categories

Markov kernels and Markov categories offer a compositional approach to probability and statistics [Čen82, Gir82, Pan99, Fon12, Fri20, Pat20].

Definition. *A* Markov kernel $M\colon \mathcal{X} \to \mathcal{Y}$ *is a measurable map* $\mathcal{X} \to \mathrm{Prob}(\mathcal{Y})$*.*

Markov kernels are "randomized functions." Markov categories axiomatize the most essential features of the category of Markov kernels.

A *Markov category* is like a cartesian category, except that some morphisms may not preserve the copying of data, representing **nondeterminism**:



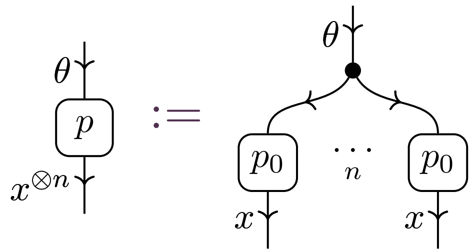
Statistical theories and models

Upgraded with extra linear algebraic structure, Markov categories allow the dictionary of categorical logic to be extended to everyday statistical models:

Category theory	Logic	Statistics
Category $\mathcal{C}$	Theory	Statistical theory
Functor $\mathcal{C} \to \mathcal{S}$	Model	Statistical model
Natural transformation	Model homomorphism	Morphism of statistical model

Definition. *A* statistical theory *is a small linear algebraic Markov category together with a distinguished morphism* $p\colon \theta \to x$*, the* sampling morphism*.*

Example. The *theory of* n *i.i.d. samples* is freely generated by a morphism $p_0\colon \theta \to x$ on discrete objects θ, x and has sampling morphism:

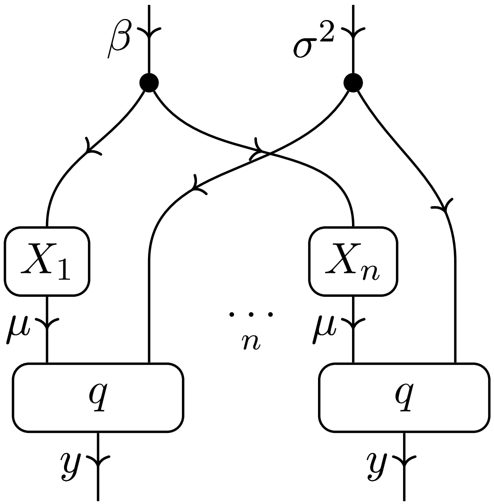


Theory of a linear model

Example. The *theory of a linear model on n observations* is presented by

- vector space objects β, μ, and y and conical space object σ^2
- linear maps $X_1, \dots, X_n \colon \beta \to \mu$
- linear-quadratic morphism $q \colon \beta \otimes \sigma^2 \to y$

and has sampling morphism $p \colon \beta \otimes \sigma^2 \to y^{\otimes n}$ given by



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