Stable Species of Structures

Marcelo Fiore
University of Cambridge

(jww Zeinab Galal & Hugo Paquet)

Workshop on Polynomial Functors
Topos Institute
19.III.2021

Background

Logic & Computation

Combinatorics & Category Theory

[Girard]
Power Series

$C \in \underline{\underline{Set}}^{\underline{Nat}}$

Generating endofunctor

$X \mapsto \sum_n C_n \times X^n$

(intensional)

[$\mathbb{L}$-species
[Joyal]]

(extensional)

[Joyal]
Species of Structure

$S \in \underline{\underline{Set}}^{\underline{Bij}}$

Generating Exponential endofunctor

$X \mapsto \sum_n S_n \times X^n / \mathfrak{S}_n$

Background

Logic & Computation

[Girard]
Power Series

$C \in \underline{\underline{Set}}^{\underline{Nat}}$

Generating endofunctor

$X \mapsto \sum_n C_n \times X^n$

Stable = finitary and wide-pullback preserving

(intensional)

[**L**-species
[Joyal]]

(extensional)

Combinatorics & Category Theory

[Joyal]
Species of Structure

$S \in Set^{\underline{\mathbb{B}ij}}$
$=$

Generating Exponential endofunctor

$X \mapsto \sum_n S_n \times X^n / G_n$

Analytic = finitary and wide quasi-pullback preserving.

(set theoretic)

X, Y sets

$$\frac{\underline{\mathrm{Set}}^X \xrightarrow{\text{stable}} \underline{\mathrm{Set}}^Y}{!(X) \times Y \to \underline{\mathrm{Set}}}$$

[Girard]

free commutative monoid

(set theoretic)

X, Y sets

$$\dfrac{\underline{Set}^X \xrightarrow{\text{stable}} \underline{Set}^Y}{!(X) \times Y \longrightarrow \underline{Set}}$$ [Girard]

$\cong$ free commutative monoid

(groupoid theoretic)

C, D creeds

$$\dfrac{\underline{QD}(C) \xrightarrow{\text{stable}} \underline{QD}(D)}{!\,\underline{QD}(C) \xrightarrow[\text{linear}]{} \underline{QD}(D)}$$ [Taylor]

involves the symmetric monoidal completion

(set theoretic)

X, Y sets

$$\frac{\mathbf{Set}^X \xrightarrow{\text{stable}} \mathbf{Set}^Y}{\;!(X) \times Y \to \mathbf{Set}\;}$$

[Girard]

free commutative monoid

(groupoid theoretic)

C, D creeds

$$\frac{\mathbf{QD}(C) \xrightarrow{\text{stable}} \mathbf{QD}(D)}{\;!\,\mathbf{QD}(C) \xrightarrow[\text{linear}]{} \mathbf{QD}(D)\;}$$

[Taylor]

involves the symmetric monoidal completion

(category theoretic)

$\mathbb{A}, \mathbb{B}$ categories

[FGHW]

symmetric monoidal completion

generalised species

$$\frac{\Sigma(\mathbb{A}) \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!+\quad\;\; \mathbb{B}}{\;P(\mathbb{A}) \longrightarrow P(\mathbb{B})\;}$$

generating exponential functor

presheaf construction

(set theoretic)

X, Y sets

$$\begin{array}{c} \underline{\underline{\mathrm{Set}}}^{X} \xrightarrow{\text{stable}} \underline{\underline{\mathrm{Set}}}^{Y} \\ \hline\hline \,!(X) \times Y \longrightarrow \underline{\underline{\mathrm{Set}}} \end{array} \quad \text{[Girard]}$$

free commutative monoid

(groupoid theoretic)

C, D creeds

$$\begin{array}{c} \underline{\underline{QD}}(C) \xrightarrow{\text{stable}} \underline{\underline{QD}}(D) \\ \hline\hline \,!\,\underline{\underline{QD}}(C) \xrightarrow[\text{linear}]{} \underline{\underline{QD}}(D) \end{array} \quad \text{[Taylor]}$$

involves the symmetric monoidal completion

(category theoretic)

$\mathbb{A}, \mathbb{B}$ categories [FGHW]

symmetric monoidal completion

generalised species

$$\begin{array}{c} \Sigma(\mathbb{A}) \longrightarrow\!\!\!\!\!\!\!\!+\quad\; \mathbb{B} \\ \hline\hline \mathcal{P}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{B}) \end{array}$$

generating exponential functor

presheaf construction

(groupoid theoretic)

$\mathbb{G}, \mathbb{H}$ groupoids

$$\begin{array}{c} \Sigma(\mathbb{G}) \longrightarrow\!\!\!\!\!\!\!\!+\quad\; \mathbb{H} \\ \hline\hline \mathcal{P}(\mathbb{G}) \xrightarrow{\text{analytic}} \mathcal{P}(\mathbb{H}) \end{array} \quad \text{[F]}$$

Direction

Logic & Computation

Combinatorics & Category Theory

A model of Differential Classical Linear Logic of stable species of structure

Direction

Logic & Computation

Combinatorics & Category Theory

A model of Differential Classical Linear Logic of stable species of structure

linear mutation of generalised species corresponding to stable functors

*-autonomous + linear exponential comonad

$\Longrightarrow$ coKleisli cartesian closure

Introductory Ideas

$\mathbb{L}$-species $\qquad\qquad\qquad\qquad$ Species

$$
\begin{tikzcd}
\underline{\underline{\mathrm{Set}}}^{\underline{\mathrm{Nat}}} \arrow[r, shift right=2] \arrow[r, phantom, "\perp"] & \mathrm{Set}^{\underline{\mathrm{Bij}}} \arrow[l, shift right=2]
\end{tikzcd}
$$

$$C \longmapsto C^{\#} : [n] \mapsto C_n \times \mathfrak{S}_n$$

with <u>free action</u>

$$C^{\#}[n] \times \mathfrak{S}_n \to C^{\#}[n]$$

$$(c, \sigma) \cdot \tau = (c, \sigma \circ \tau)$$

$$(c, \sigma) \cdot \tau = (c, \sigma) \Rightarrow \tau = \mathrm{id}$$

Introductory Ideas

$\mathbb{L}$-species $\qquad\qquad\qquad\qquad$ Species

$$\begin{tikzcd} \underline{\mathrm{Set}}^{\underline{\mathbb{N}\mathrm{at}}} \arrow[r, shift right, phantom, "\lambda"] \arrow[r, shift right=2] & \mathrm{Set}^{\underline{\mathbb{B}\mathrm{ij}}} \arrow[l, shift right=2] \end{tikzcd}$$

$$C \longmapsto C^{\#} : [n] \mapsto C_n \times \mathfrak{S}_n$$

with <u>free action</u>

$$C^{\#}[n] \times \mathfrak{S}_n \to C^{\#}[n]$$

$$(c, \sigma) \cdot \tau = (c, \sigma \circ \tau)$$

$$(c,\sigma) \cdot \tau = (c,\sigma) \Rightarrow \tau = \mathrm{id}$$

$$\underline{\mathrm{Stab}}(c,\sigma) = \{ \tau \mid (c,\sigma)\cdot\tau = (c,\sigma) \} = \{ \mathrm{id} \}$$

$S \in \underline{\underline{\mathrm{Set}}}^{\mathbb{B}ij}$ is an $\mathbb{L}$-species

iff

$\forall\, s \in S(n)$:

$$\underline{\mathrm{Stab}}(s) = \{ \tau \in \mathfrak{S}_n \mid s \cdot \tau = s \} = \{ \mathrm{id} \}$$

$\mathbb{L}$-species $\longleftrightarrow$ Species

[Taylor]

consider degrees of stabilisation

$S \in \underline{\underline{\mathrm{Set}}}^{\mathrm{Bij}}$

- Stabilisers form subgroups

$$\underline{\mathrm{Stab}}(s) = \{ \tau \in \mathfrak{S}_n \mid s \cdot \tau = s \} \leq \mathfrak{S}_n$$

- Stabiliser subgroups are closed under conjugation

$$\underline{\mathrm{Stab}}(s \cdot \sigma) = \sigma^{-1}\, \underline{\mathrm{Stab}}(s)\, \sigma$$

- Families of subgroups closed under conjugation

$$\underline{\mathrm{Stab}}[n] = \{ \mathrm{Stab}(s) \leq \mathfrak{S}_n \mid s \in S[n] \}$$

Def A kit on a groupoid $\mathbb{A}$ is an assignment $\mathcal{A}$

$$a \in \mathbb{A} \longmapsto \mathcal{A}(a) \subseteq \underline{\mathrm{SubGp}}\left(\mathrm{Aut}_{\mathbb{A}}(a)\right)$$

a family of subgroups of the group of automorphisms on a in $\mathbb{A}$.

closed under conjugation

$$\begin{array}{ccc} a & & G \in \mathcal{A}(a) \\ \alpha \big\downarrow & & \Downarrow \\ a' & & \alpha G \alpha^{-1} \in \mathcal{A}(a') \end{array}$$

Combinatorial Character of Kits

Def: Stabilised presheaves

$$\mathcal{S}(A, \mathscr{A}) \hookrightarrow \mathcal{P}(A) = \mathcal{S}et^{A^{o}}$$

full subcategory of presheaves P such that

$$\underline{\mathrm{Stab}}_P(p) \in \mathscr{A}(a) \quad \forall a \ \forall p$$

Combinatorial Character of Kits

Def: Stabilised presheaves

$$\mathcal{S}(\mathbb{A},\mathscr{A}) \hookrightarrow \mathcal{P}(\mathbb{A}) = \mathbf{Set}^{\mathbb{A}^{\circ}}$$

full subcategory of presheaves P such that

$$\underline{\mathrm{Stab}}_P(p) \in \mathscr{A}(a) \quad \forall a \ \forall p$$

Examples:

- $1 \in \mathcal{S}(\mathbb{A},\mathscr{A}) \iff \underline{\mathrm{Aut}}_{\mathbb{A}}(a) \in \mathscr{A}(a)$
- $\mathcal{S}(\underline{\mathbb{B}\mathrm{ij}}, \mathrm{Id}) \longrightarrow \mathcal{S}(\underline{\mathbb{B}\mathrm{ij}}, \underline{\mathrm{SubGr}})$ EM-algs

  $\mathcal{S}(\underline{\mathbb{B}\mathrm{ij}}, \mathrm{Id}) \cong (\underline{\underline{\mathbf{Set}}}^{\mathbb{N}\mathrm{at}})_{\#}$ Kleisli = free algs

  $\mathcal{S}(\underline{\mathbb{B}\mathrm{ij}}, \underline{\mathrm{SubGr}}) = \underline{\underline{\mathbf{Set}}}^{\mathbb{B}\mathrm{ij}} \cong (\underline{\underline{\mathbf{Set}}}^{\mathbb{N}\mathrm{at}})^{\#}$

Thm: Representation

$S(\mathbb{A}, \mathcal{A})$ arises as the coproduct completion of the quotients

$$\left\{ y(a)/G \right\}_{a \in \mathbb{A},\, G \in \mathcal{A}(a)}$$

Kits with logical character

~ the Boolean algebra of a group ~

Kits with logical character

A group

$$\underline{\mathrm{SubGr}}(A) \hookrightarrow \mathcal{F}_{free}(A) = \left[\underline{\mathrm{SubGr}}(A), \begin{pmatrix} \top \\ \vee\!\!\mid \\ \bot \end{pmatrix}\right]$$

free frame

(Heyting alg)

Kits with logical character

A group

$$\begin{tikzcd}[column sep=large, row sep=large]
\underline{\mathrm{SubGr}}(A) \arrow[r, hook] \arrow[dr, bend right] & \mathcal{F}_{\mathrm{free}}(A) \arrow[d, bend right] \arrow[d, phantom, "\perp"] \\
& \mathcal{F}_{\mathrm{can}}(A) \arrow[u, bend right]
\end{tikzcd}$$

$\mathcal{F}_{\mathrm{free}}(A) = \left[\underline{\mathrm{Sub}}\,\underline{\mathrm{Gr}}(A), \begin{pmatrix}\top \\ \rotatebox[origin=c]{90}{$\leq$} \\ \bot\end{pmatrix}\right]$ free frame

$\mathcal{F}_{\mathrm{can}}(A)$ canonical frame (Heyting algs)

Kits with logical character

~ the Boolean algebra of a group ~

A group

$$
\begin{tikzcd}
\underline{\mathrm{SubGr}}(A) \arrow[r, hook] \arrow[dr, bend right] & \mathcal{F}_{free}(A) \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \\
 & \mathcal{F}_{can}(A) \arrow[u, bend right] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \\
 & \big(\mathcal{F}_{can}(A)\big)_{\neg\neg} \arrow[u, bend right]
\end{tikzcd}
$$

$\mathcal{F}_{free}(A) = \left[\underline{\mathrm{SubGr}}(A), \begin{pmatrix}\top \\ \geq \\ \bot\end{pmatrix}\right]$ free frame

$\mathcal{F}_{can}(A)$ canonical frame (Heyting algs)

$\big(\mathcal{F}_{can}(A)\big)_{\neg\neg} = \mathcal{B}(A)$ (Booleanisation)

Logic	Truth values
$\mathcal{F}_{free}(A)$	downwards closed families
$\mathcal{F}_{Scott}(A)$	directed families
$\mathcal{F}_{Taylor}(A)$	extensional families (creeds) $(\mathcal{A} = \{G \leq A \mid G \subseteq \bigcup \mathcal{A}\})$
$\mathcal{F}_{con}(A)$

$$\frac{\text{intuitionistic}}{\text{classical}}$$

$\mathcal{B}(A)$ — double-orthogonal families

Def : Orthogonality

For $G \leq A$ and $H \leq A^{\circ}$,

$$G \perp H \iff G \cap H = \{id\}$$

Def: Orthogonality

For $G \leq A$ and $H \leq A^\circ$,

$$G \perp H \iff G \cap H = \{id\}$$

Def: Orthogonal complement

For $\mathcal{A} \subseteq \underline{\mathrm{SubGr}}(A)$,

$$\mathcal{A}^\perp = \{ H \leq A^\circ \mid \forall G \in \mathcal{A}.\ G \perp H \}$$

Def: Orthogonality

For $G \leqslant A$ and $H \leqslant A^\circ$,

$$G \perp H \iff G \cap H = \{id\}$$

Def: Orthogonal complement

For $\mathcal{A} \subseteq \underline{\mathrm{SubGr}}(A)$,

$$\mathcal{A}^\perp = \{ H \leqslant A^\circ \mid \forall G \in \mathcal{A}.\ G \perp H \}$$

Thm:

$$\mathcal{B}(A) = \{ \mathcal{A} \subseteq \underline{\mathrm{SubGr}}(A) \mid \mathcal{A} = \mathcal{A}^{\perp\perp} \}$$

P.S. Examples:

- $\mathcal{B}(C_6) \cong 2^2$
- $\mathcal{B}(C_{12}) \cong 2^2$
- $\mathcal{B}(S_3) \cong 2^4$

- $\mathcal{X} \subseteq \mathcal{X}^{\perp\perp}$

- $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{Y}^{\perp} \subseteq \mathcal{X}^{\perp}$

- $\underbrace{\emptyset^{\perp} = \mathrm{SubGr}(A)}_{\mathrm{True}}$ $\qquad$ $\underbrace{\mathrm{True}^{\perp} = \mathrm{Id}}_{\mathrm{False}}$

- $P \wedge Q = P \cap Q$

- $P \vee Q = (P \cup Q)^{\perp\perp}$

Def: A Boolean kit $\mathcal{A}$ on a groupoid A is a kit such that

$$\forall a \in A. \quad \mathcal{A}(a) = \mathcal{A}(a)^{\perp\perp}$$

that is,

$\mathcal{A}(a)$ is a truth value in the Boolean algebra associated to $\underline{\mathrm{Aut}}_A(a)$.

Prop: If $\mathcal{A}$ is a Boolean kit on $\mathbb{A}$, Then $\mathcal{A}^{\perp} = \{\mathcal{A}(a)^{\perp}\}_{a \in \mathbb{A}}$ is a Boolean kit on $\mathbb{A}^{\circ}$.

P.S.

$$\overline{\mathcal{B}}(A) = \{ \mathcal{A} \subseteq \underline{\mathrm{SubGr}}(A) \mid \mathcal{A} \text{ is closed under conjugation}, \ \mathcal{A}^{\perp\perp} = \mathcal{A} \}$$

is a sub Boolean algebra of $\mathcal{B}(A)$

Examples:

- $\overline{\mathcal{B}}(C_6) \cong \overline{\mathcal{B}}(C_{12}) \cong 2^2$
- $\overline{\mathcal{B}}(S_3) \cong 2^2$
- $\overline{\mathcal{B}}(S_4) \cong 2^3$

Amalgamating Dualities

Prop: If $\mathscr{A}$ is a Boolean Rif on $\mathbb{A}$, then $\mathscr{A}^\perp = \{\mathscr{A}(a)^\perp\}_{a \in \mathbb{A}}$ is a Boolean Rif on $\mathbb{A}^\circ$

- We have a duality

$$(\mathbb{A}, \mathscr{A})^* = (\mathbb{A}^\circ, \mathscr{A}^\perp)$$

Properties

- Boolean kits are downward and directed closed

<u>Thm</u>: Representation

For a Boolean kit $(A, \mathcal{A})$,

$$\mathcal{S}(A, \mathcal{A}) \simeq \underline{\mathrm{Ind}}\left(\underline{\mathrm{Fam}}_{fin}\left(\{ y(a)/G \}_{\substack{G \text{ finite} \\ \text{in } \mathcal{A}(a)}}\right)\right)$$

Properties

- Boolean kits are downward and directed closed
- Boolean kits are extensional (i.e. creeds [Taylor])

$$\mathcal{A}(a) = \{ G \leq \underline{\mathrm{Aut}}_{/\!\!A}(a) \mid G \subseteq \bigcup \mathcal{A} \}$$

Notation:

$$\alpha \Vdash \mathcal{A}(a) \iff \alpha \in \bigcup \mathcal{A} \qquad \left(\alpha \in \underline{\mathrm{Aut}}(a)\right)$$

Boolean kit logic

- $\mathcal{A} \subseteq \mathcal{B} \iff [\forall \alpha.\ \alpha \Vdash \mathcal{A} \Rightarrow \alpha \Vdash \mathcal{B}]$
- $\alpha \Vdash \mathrm{True}$
- $\alpha \Vdash \mathrm{False} \iff \alpha = \mathrm{id}$
- $\alpha \Vdash \mathcal{A} \iff \forall n.\ \alpha^n \Vdash \mathcal{A}$
- $$\frac{\alpha \Vdash \mathcal{A} \qquad \alpha \Vdash \mathcal{A}^{\perp}}{\alpha \Vdash \mathrm{False}}$$

A linear setting for Boolean kits

from profunctors between groupoids

to

stabilized profunctors between Boolean kits

A linear setting for Boolean kits

from profunctors between groupoids
to
stabilized profunctors between Boolean kits

Desiderata I:

$$\frac{(A,\mathcal{A}) \xrightarrow{P} (B,\mathcal{B})}{(A,\mathcal{A})^* \xleftarrow{P^*} (B,\mathcal{B})^*}$$

duality

A linear setting for Boolean kits

from profunctors between groupoids
to
stabilized profunctors between Boolean kits

Desiderata II :

$$\frac{\mathcal{A} \subseteq \mathcal{B}}{(\mathbb{C}, \mathcal{A}) \xrightarrow[\text{hom}]{\quad} (\mathbb{C}, \mathcal{B})}$$

Boolean fibers

Def: A stabilised profunctor (s-profunctor) $(\mathbb{A},\mathcal{A}) \nrightarrow (\mathbb{B},\mathcal{B})$ between Boolean kits is a profunctor $P : \mathbb{A} \nrightarrow \mathbb{B}$ such that

$$\begin{tikzcd} & \cdot \arrow[r, squiggly, "p"] & \cdot \arrow[dr, "\alpha"] & \\ \cdot \arrow[ur, "\beta"] \arrow[rrr, squiggly, "p"'] & & & \cdot \end{tikzcd}$$

$p \in P(b,a)$

$\alpha \in \underline{Aut}_{\mathbb{A}}(a)$

$\beta \in \underline{Aut}_{\mathbb{B}}(b)$

implies

$$\left(\alpha \Vdash \mathcal{A} \Rightarrow \beta \Vdash \mathcal{B}\right) \text{ and } \left(\beta \Vdash \mathcal{B}^{\perp} \Rightarrow \alpha \Vdash \mathcal{A}^{\perp}\right)$$

Prop: Desiderata I (duality) and Desiderata II (Boolean fibers) hold

Examples:

- Every profunctor $A \nrightarrow B$ is a stabilized profunctor $(A, \underline{\text{False}}) \nrightarrow (B, \underline{\text{True}})$.

Prop: Desiderata $\underline{\mathrm{I}}$ (duality) and Desiderata $\underline{\mathrm{II}}$ (Boolean fibers) hold

Examples:

- Every profunctor $\mathbb{A} \nrightarrow \mathbb{B}$ is a stabilised profunctor $(\mathbb{A}, \underline{\mathrm{False}}) \nrightarrow (\mathbb{B}, \underline{\mathrm{True}})$.
- Stabilised profunctors $(\mathbb{A}, \underline{\mathrm{True}}) \nrightarrow (\mathbb{B}, \underline{\mathrm{True}})$ are profunctors with a free $\mathbb{A}$-action.

Def: $\underline{\mathbf{SProf}}$ is the bicategory of

- Boolean kits
- S-profunctors
- natural transformations

$$\begin{tikzcd} \underline{\mathbf{SProf}} \arrow[d] \\ \underline{\mathbf{Prof}}_{gpd} \end{tikzcd}$$

Def: $\underline{\underline{\mathrm{SProf}}}$ is the bicategory of

- Boolean kits
- s-profunctors
- natural transformations

$$\begin{tikzcd} \underline{\underline{\mathrm{SProf}}} \arrow[d] \\ \underline{\underline{\mathrm{Prof}}}_{gpd} \end{tikzcd}$$

Thm: The biproduct and compact closed structure of $\underline{\underline{\mathrm{Prof}}}_{gpd}$ lifts to a biproduct and $*$-autonomous structure on $\underline{\underline{\mathrm{SProf}}}$

Biproduct structure

$$\bigoplus_i (A_i, \mathcal{A}_i) = \left(\coprod_i A_i , \coprod_i \mathcal{A}_i \right)$$

$$\iota_k(\alpha) \Vdash \left(\coprod_i \mathcal{A}_i \right) (\iota_k a) \iff \alpha \Vdash \mathcal{A}_k(a)$$

Symmetric Monoidal Structure

unit

$(\mathbb{1}, \mathcal{I})$ $\qquad\qquad \mathcal{I}(*) = \{\{id\}\}$

tensor product

$$(A, \mathcal{A}) \otimes (B, \mathcal{B}) = (A \times B, (\mathcal{A} \times \mathcal{B})^{\perp\perp})$$

Thm: The structural profunctors are stabilised

Thm: The structural profunctors are stabilised

For the associator this is induced by the logical rule:

$$\frac{(x, y) \Vdash (\mathcal{X} \times y)^{\perp} \qquad x \Vdash \mathcal{X}}{y \Vdash y^{\perp}}$$

*-autonomous structure

$$\underline{\mathrm{SProf}}\left((\mathbb{A},\mathscr{A}) \otimes (\mathbb{B},\mathscr{B}) \,,\, (\mathbb{C},\mathscr{C})^{*} \right)$$

$$\cong \quad \underline{\mathrm{SProf}}\left((\mathbb{B},\mathscr{B}) \,,\, \left((\mathbb{C},\mathscr{C}) \otimes (\mathbb{A},\mathscr{A})\right)^{*} \right)$$

because

$$\frac{\mathbb{A} \times \mathbb{B} \longrightarrow\!\!\!\!\!\!+\;\; \mathbb{C}^{o}}{\mathbb{B} \longrightarrow\!\!\!\!\!\!+\;\; (\mathbb{C} \times \mathbb{A})^{o}} \qquad \Longleftrightarrow \qquad \begin{array}{l} \text{s-profunctor} \\ \text{s-profunctor} \end{array}$$

$\underline{\mathbf{SProf}}$ is not compact closed

$$(\mathbb{A}, \mathcal{A}) \parr (\mathbb{B}, \mathcal{B}) = \left((\mathbb{A}, \mathcal{A})^* \otimes (\mathbb{B}, \mathcal{B})^* \right)^*$$

$$= \left(\mathbb{A} \times \mathbb{B}, (\mathcal{A}^\perp \times \mathcal{B}^\perp)^\perp \right)$$

SProf is not compact closed

$$(\mathbb{A}, \mathcal{A}) \parr (\mathbb{B}, \mathcal{B}) = \left((\mathbb{A}, \mathcal{A})^* \otimes (\mathbb{B}, \mathcal{B})^* \right)^*$$

$$= \left(\mathbb{A} \times \mathbb{B}, (\mathcal{A}^\perp \times \mathcal{B}^\perp)^\perp \right)$$

Closed Structure

$$(\mathbb{A}, \mathcal{A}) \multimap (\mathbb{B}, \mathcal{B}) = (\mathbb{A}, \mathcal{A})^* \parr (\mathbb{B}, \mathcal{B})$$

$$= \left(\mathbb{A}^0 \times \mathbb{B}, (\mathcal{A} \times \mathcal{B}^\perp)^\perp \right)$$

A combinatorial example:

$$(\mathbb{B}ij, \underline{True}) \multimap (\mathbb{B}ij, \underline{True})$$

$$= \left(\underline{\mathbb{B}ij}^{\circ} \times \underline{\mathbb{B}ij} \;,\; (\underline{True} \times \underline{False})^{\perp} \right)$$

A combinatorial example:

$$(\underline{\mathbb{B}ij}, \underline{True}) \multimap (\underline{\mathbb{B}ij}, \underline{True})$$

$$= \left(\underline{\mathbb{B}ij}^{\circ} \times \underline{\mathbb{B}ij} \,,\, (\underline{True} \times \underline{False})^{\perp} \right)$$

$$(\sigma, \tau) \Vdash (True \times False)^{\perp}$$

iff

$$\underline{order}(\tau) \text{ divides } \underline{order}(\sigma)$$

Exponential structure

Exponential structure

- The symmetric monoidal completion for groupoids
$$\mathbb{A} \longmapsto \Sigma(\mathbb{A})$$
lifts to a linear exponential comonad on $\mathbf{Prof}_{\mathrm{gpd}}$ [FGHW]

$$\Sigma(\mathbb{A}+\mathbb{B}) \simeq \Sigma(\mathbb{A}) \times \Sigma(\mathbb{B})$$

Exponential structure

- The symmetric monoidal completion for groupoids
$$\mathbb{A} \longmapsto \Sigma(\mathbb{A})$$
lifts to a <u>linear exponential comonad</u> on $\mathbf{Prof}_{gpd}$ [FGHW]
$$\Sigma(\mathbb{A}+\mathbb{B}) \simeq \Sigma(\mathbb{A}) \times \Sigma(\mathbb{B})$$

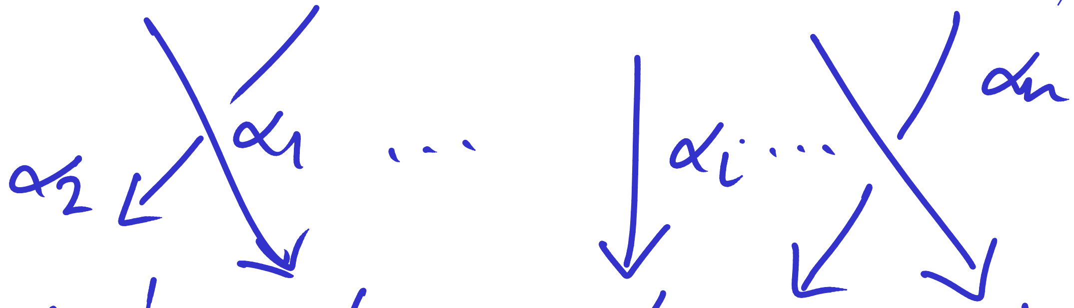
- We lift it to $\mathbf{SProf}$:
$$!(\mathbb{A}, \mathcal{A}) = (\Sigma(\mathbb{A}), !\mathcal{A})$$

Symmetric-Monoidal Completion

$\Sigma(\mathbb{A})$ objects: $(a_1, \ldots, a_n)$ $n \in \mathbb{N}$

morphisms: $(a_1, a_2, \ldots, a_i, \ldots, a_n)$

$\sigma \in \mathfrak{S}_n$

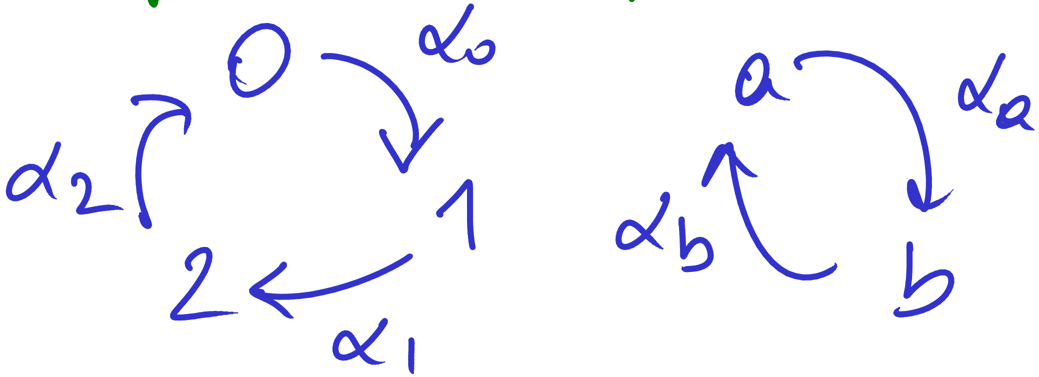


$(a'_1, a'_2, \ldots, a'_{\sigma i}, \ldots, a'_n)$

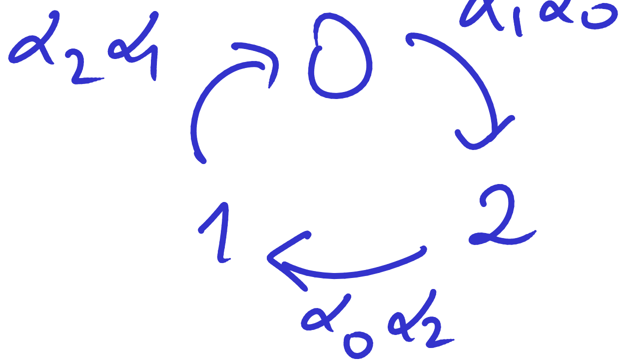
$$
\begin{tikzcd}
 & & & 0 \arrow[dr, bend left, "\alpha_0"] & & a \arrow[d, bend left, "\alpha_a"] \\
(0,1,2,a,b) \arrow[loop right, "\alpha"] & = & 2 \arrow[ur, bend left, "\alpha_2"] & & 1 \arrow[ll, bend left, "\alpha_1"] & b \arrow[u, bend left, "\alpha_b"]
\end{tikzcd}
$$

Endomorphism Loops

$(0,1,2,a,b) \circlearrowleft \alpha =$



$\alpha^2 =$

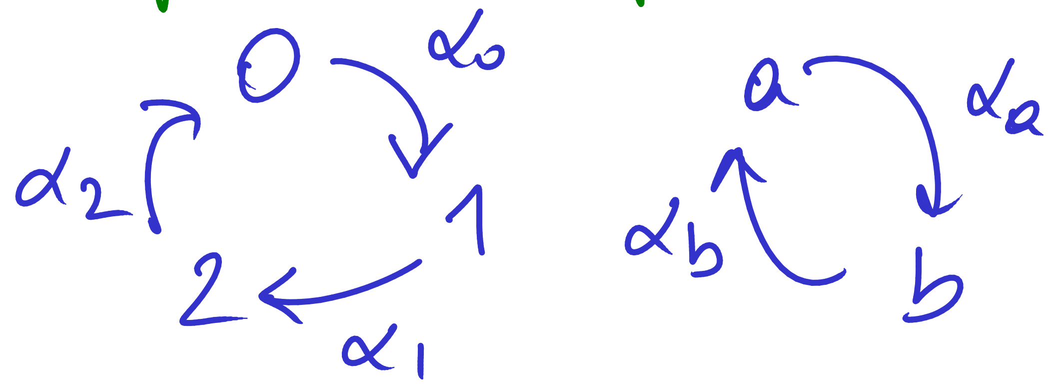


$a \circlearrowleft \ell_\alpha(a) = \alpha_b \alpha_a$

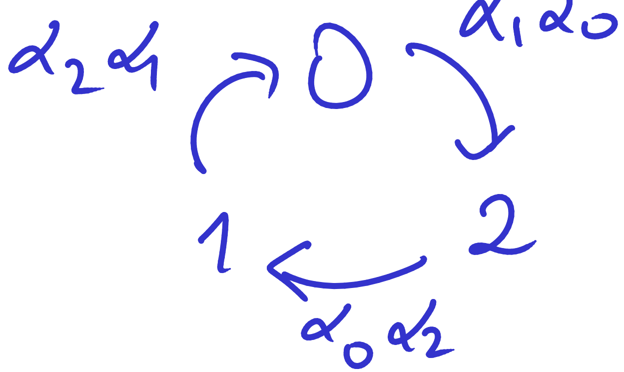
$b \circlearrowleft \ell_\alpha(b) = \alpha_a \alpha_b$

Endomorphism Loops

$(0,1,2,a,b)\,\alpha =$



$\alpha^2 =$



a $\ell_\alpha(a) = \alpha_b\alpha_a$

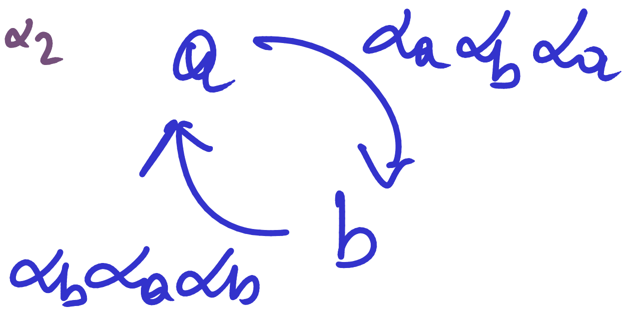
b $\ell_\alpha(b) = \alpha_a\alpha_b$

$\alpha^3 =$

$\alpha_2\alpha_1\alpha_0 = \ell_\alpha(0)$ 0

$\ell_\alpha(1) = \alpha_0\alpha_1\alpha_2$ 1

2 $\ell_\alpha(2) = \alpha_1\alpha_0\alpha_2$



Def:
$$!(\mathbb{A}, \mathcal{A}) = (\Sigma(\mathbb{A}), !\mathcal{A})$$

$$!\mathcal{A}(a_1, \dots, a_n)$$
$$= \{ \alpha \in \mathrm{Aut}(a_1 - a_n) \mid \forall i.\ \ell_\alpha(a_i) \in \mathcal{A}(a_i) \}^{\perp\perp}$$

Example:
$$!(\mathbb{A}, \mathrm{True}) = (\Sigma(\mathbb{A}), \mathrm{True})$$

Thm:

$$\underline{\mathrm{SProf}}, \otimes, (-)^*, !$$

is a bicategorical model of differential classical linear logic.

Stable Species of Structures

Cor: The coKleisli bicategory

$$\underline{\mathbf{SEsp}} = \underline{\mathbf{SProf}}_!$$

is cartesian closed and cartesian differential.

$$(\mathbb{A},\mathscr{A}) \Rightarrow (\mathbb{B},\mathscr{B})$$

$$= !(\mathbb{A},\mathscr{A}) \multimap (\mathbb{B},\mathscr{B})$$

$$= (\Sigma \mathbb{A}^{\circ} \times \mathbb{B}, (!\mathscr{A} \times \mathscr{B}^{\perp})^{\perp})$$

Stable Species of Structures

Cor: The coKleisli bicategory

$$\underline{\underline{\mathrm{SEsp}}} = \underline{\mathrm{SProf}}_!$$

is cartesian closed and cartesian differential.

$$(\mathbb{A},\mathcal{A}) \Rightarrow (\mathbb{B},\mathcal{B})$$
$$= !(\mathbb{A},\mathcal{A}) \multimap (\mathbb{B},\mathcal{B})$$
$$= (\Sigma\mathbb{A}^{\circ}\times\mathbb{B}, (!\mathcal{A}\times\mathcal{B}^{\perp})^{\perp})$$

Example: $\underline{\mathrm{SEsp}}((\mathbb{1},\mathcal{I}),(\mathbb{1},\mathcal{I})) \simeq (\mathrm{Set}^{\mathrm{Nat}})_{\#}$ $\mathbb{L}$-species

Stable Species of Structures

Cor: The coKleisli bicategory

$$\underline{\underline{SEsp}} = S\underline{\mathbf{Prof}}_!$$

is cartesian closed and cartesian differential.

$$(\mathbb{A}, \mathcal{A}) \Rightarrow (\mathbb{B}, \mathcal{B})$$
$$= !(\mathbb{A}, \mathcal{A}) \multimap (\mathbb{B}, \mathcal{B})$$
$$= (\Sigma \mathbb{A}^{\circ} \times \mathbb{B}, (!\mathcal{A} \times \mathcal{B}^{\perp})^{\perp})$$

Example: $\underline{SEsp}((\mathbb{1}, I), (\mathbb{1}, I)) \simeq (\mathrm{Set}^{\mathbb{N}\mathrm{at}})_{\#}$ $\mathbb{L}$-species

Remark:

$$\begin{array}{cc} \underline{SEsp} & (\mathbb{A}, \mathcal{A}) \\ \downarrow & \downarrow \\ \underline{Esp}\,\mathrm{gpd} & \mathbb{A} \end{array}$$

preserves cartesian closed and differential structure

Generating Functors

$P \in \underline{\mathrm{SEsp}}((\mathbb{A}, \mathscr{A}), (\mathbb{B}, \mathscr{B}))$

$$\begin{tikzcd}
\Sigma(\mathbb{A}) \arrow[r, "\mathrm{sum}"] \arrow[dr, "P"'] & \underline{S}(\mathbb{A}, \mathscr{A}) \arrow[r, hook] \arrow[d, phantom, "\overset{\mathrm{Lan}}{\Longrightarrow}"] & \mathcal{P}(\mathbb{A}) \arrow[d, "\widetilde{P}"] \\
& \underline{S}(\mathbb{B}, \mathscr{B}) \arrow[r, hook] & \mathcal{P}(\mathbb{B})
\end{tikzcd}$$

analytic

$$\widetilde{P}(X) = \int^{(a_1 \ldots a_n) \in \Sigma(\mathbb{A})} P(a_1, \ldots a_n) \cdot \prod_{i=1}^{n} X(a_i)$$

Generating Functors

$P \in \underline{\underline{\mathrm{SEsp}}}\big((\mathbb{A},\mathcal{A}),(\mathbb{B},\mathcal{B})\big)$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\Sigma(\mathbb{A}) \arrow[r, "\mathrm{sum}"] \arrow[dr, "P"', ""{name=P}] & \underline{S}(\mathbb{A},\mathcal{A}) \arrow[r, hook] \arrow[d] \arrow[dl, phantom, "\overset{\mathrm{Lan}}{\Longrightarrow}" description, pos=0.3] & \mathcal{P}(\mathbb{A}) \arrow[d, "\widetilde{P}"] \\
 & \underline{S}(\mathbb{B},\mathcal{B}) \arrow[r, hook] & \mathcal{P}(\mathbb{B})
\end{tikzcd}
$$

$\underline{\underline{\text{stable}}}$: finitary
epi-preserving
local right adjoint

analytic

$$
\widetilde{P}(X) = \int^{(a_1 \ldots a_n) \in \Sigma(\mathbb{A})} P(a_1, \ldots, a_n) \cdot \prod_{i=1}^{n} X(a_i)
$$

Intensional/Extensional Biequivalence

Thm:

$$\mathbf{SEsp} \simeq \mathbf{Stable}$$

bicat

- Boolean kits
- stable species
- natural transformations

2-cat

- Boolean kits
- stable functors between stabilised presheaves
- cartesian natural transformations

finitary epi-preserving local right adjoints

Thm:

$$\underline{\underline{\mathrm{SProf}}} \simeq \underline{\underline{\mathrm{Linear}}}$$

bicat

- Boolean kits
- stabilised profunctors
- natural transformations

2-cat

- Boolean kits
- Linear functors between stabilised presheaves
- cartesian natural transformations

local left and right adjoints

Linear Decomposition

<u>**Thm**</u>:

$$\underline{\underline{\mathrm{Stable}}}\big((\mathbb{A},\mathcal{A}),(\mathbb{B},\mathcal{B})\big) \simeq \underline{\underline{\mathrm{Linear}}}\big(!(\mathbb{A},\mathcal{A}),(\mathbb{B},\mathcal{B})\big)$$

References

* M Barr. *-Autonomous categories. Lecture Notes in Mathematics, 1979.

* A Joyal. Une theorie combinatoire des series formelles. Advances in Mathematics, 1981.

* A Joyal. Foncteurs analytiques et especes de structures. Lecture Notes in Mathematics, 1986.

* J-Y Girard. Normal functors, power series and lambda calculus. Annals of Pure and Applied Logic, 1988.

* P Taylor. Quantitative domains, groupoids and linear logic. Lecture Notes in Computer Science, 1989.

* F Bergeron, G Labelle, and P Leroux. Combinatorial species and tree-like structures. Cambridge University Press, 1998.

* J Benabou. Distributors at work. Lecture Notes, 2000.

* M Fiore. Mathematical models of computational and combinatorial structures. Lecture Notes in Computer Science, 2005.

* M Hyland and A Schalk. Glueing and orthogonality for models of linear logic. Theoretical Computer Science, 2003.

* M Fiore. Differential structure in models of multiplicative biadditive intuitionistic linear logic. Lecture Notes in Computer Science, 2007.

* M Fiore, N Gambino, M Hyland, and G Winskel. The cartesian closed bicategory of generalised species of structure. Journal of the London Mathematical Society, 2008.

* R Blute, R Cockett, and R Seely. Cartesian differential categories. Theory and Applications of Categories, 2009.

* M Fiore. Analytic functors between presheaf categories over groupoids. Theoretical Computer Science, 2014.

* M Fiore. An Axiomatics and a Combinatorial Model of Creation/Annihilation Operators. Arxiv, 2015.

* T Ehrhard. An introduction to differential linear logic: proof nets, models, antiderivatives. Arxiv, 2016.

* M Fiore. An algebraic combinatorial approach to opetopic structure. MPIM Seminar, 2016.

* M Stay. Compact closed bicategories. Theory and Applications of Categories, 2016.

* M Fiore, N Gambino, M Hyland, and G Winskel. Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. Selecta Mathematica, 2018.

* R Blute, R Cockett, J-S Pacaud Lemay, R Seely. Differential categories revisited. Applied Categorical Structures, 2020.

APPENDIX

Connection to Double Glueing

[Hyland & Schalk]

- Presheaf orthogonality

$$\perp \subseteq \mathcal{P}(\mathbb{A}) \times \mathcal{P}(\mathbb{A}^{o})$$

$$X \perp X' \iff X(-) \times X'(-) : \mathbb{A} \to \underline{\underline{Set}} \text{ is free}$$

$$\left\{ \begin{array}{c} S \hookrightarrow \mathcal{P}(\mathbb{A}) \\ S = S^{\perp\perp} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Boolean kits} \\ \text{on } \mathbb{A} \end{array} \right\}$$

$$\mathcal{S}(\mathbb{A}, \mathcal{A}) \longleftarrow\!\!\shortmid \mathcal{A}$$

$$P : (\mathbb{A}, \mathcal{A}) \relbar\joinrel\mapstochar\joinrel\rightarrow (\mathbb{B}, \mathcal{B}) \quad \text{an s-profunctor}$$

iff

$$
\begin{tikzcd}
S(\mathbb{A}, \mathcal{A}) \arrow[r, dashed] \arrow[d, hook] & S(\mathbb{B}, \mathcal{B}) \arrow[d, hook] \\
\mathcal{P}(\mathbb{A}) \arrow[r, "\overline{P}"'] & \mathcal{P}(\mathbb{B})
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
S(\mathbb{B}^{\circ}, \mathcal{B}^{\perp}) \arrow[r, dashed] \arrow[d, hook] & S(\mathbb{A}^{\circ}, \mathcal{A}^{\perp}) \arrow[d, hook] \\
\mathcal{P}(\mathbb{B}^{\circ}) \arrow[r, "\overline{P^{*}}"'] & \mathcal{P}(\mathbb{A}^{\circ})
\end{tikzcd}
$$

Differential Structure

- $P \in \underline{\underline{\mathrm{SEsp}}}\left((\mathbb{A},\mathcal{A}), (\mathbb{B},\mathcal{B}) \right)$

$\partial P \in \underline{\underline{\mathrm{SEsp}}}\left((\mathbb{A},\mathcal{A}), (\mathbb{A},\mathcal{A}) \multimap (\mathbb{B},\mathcal{B}) \right)$

$\partial P\left((a,b), s \right) = P(b, \langle a \rangle \oplus s)$

- $!$ is a model of Fock space with operators

$$\begin{tikzcd}[column sep=huge]
!(\mathbb{A},\mathcal{A}) \otimes (\mathbb{A},\mathcal{A}) \arrow[r, shift left, "\text{creation}"] & (\mathbb{A},\mathcal{A}) \arrow[l, shift left, "\text{annihilation}"]
\end{tikzcd}$$

satisfying commutation relations

- $!(\mathbb{A},\mathcal{A})$ is a commutative bialgebra

$$\begin{tikzcd}
!(\mathbb{A},\mathcal{A}) \arrow[r] \arrow[dr, "!\Delta"'] & !(\mathbb{A},\mathcal{A}) \otimes !(\mathbb{A},\mathcal{A}) \arrow[r] & !(\mathbb{A},\mathcal{A}) \\
& !((\mathbb{A},\mathcal{A}) \oplus (\mathbb{A},\mathcal{A})) \arrow[u, "\cong"'] \arrow[ur, "!\nabla"'] &
\end{tikzcd}$$

- Multiplication by convolution

$$
\begin{tikzcd}
 & !(\mathbb{A},\mathcal{A}) \otimes !(\mathbb{A},\mathcal{A}) \arrow[r, "P \otimes Q", "\shortmid"{marking}] & (\mathbb{B},\mathcal{B}) \otimes (\mathbb{B},\mathcal{B}) \arrow[dr, "\shortmid"{marking}] & \\
!(\mathbb{A},\mathcal{A}) \arrow[ur, "\shortmid"{marking}] \arrow[rrr, "P \bullet Q"', "\shortmid"{marking}] & & & (\mathbb{B},\mathcal{B})
\end{tikzcd}
$$