Initial Algebras in Homotopy Type Theory

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Introduction

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- Inductive types are ubiquitous in computer science: natural numbers, lists, trees...

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- They know initial algebras exist in general (under some conditions)
- Inductive types provide an explicit description of these
- Type theory gives a way to reason syntactically about categorical constructs (internal languages)

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Initial algebras and inductive types:

- Coincide in extensional type theory (Dybjer '96)
- Do *not* coincide in intensional type theory
- Coincide in homotopy type theory after replacing initiality by *homotopy-initiality* (Awodey, Gambino, S. '12)

Outline

1. Introduction
2. Extensional type theory
3. Well-founded trees
4. Initial algebras are well-founded trees (and vice versa)
5. Homotopy type theory
6. Homotopy-initial algebras
7. Homotopy-initial algebras = well-founded trees
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Extensional Type Theory

Dependent type theory (Martin-Löf) has:

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- Equality judgements: $A = B$ and $a =_A b$ for $a, b : A$

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- Equality judgements: $A = B$ and $a =_A b$ for $a, b : A$
- Identity reflection: equal types and terms are treated as identical

Intended as a foundation for constructive mathematics.

The traditional set interpretation

Suppose we have terms of ascending identity types:

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\begin{aligned}
a,\ b &: A \\
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We have the following interpretation into sets:

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\text{Terms} &\quad\rightsquigarrow\quad \text{Elements} \\
a : A &\quad\rightsquigarrow\quad \text{Element } a \in A \\
p : a =_A b &\quad\rightsquigarrow\quad \text{Element of a singleton set} \\
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Well-founded trees

Inductive types: *"structures freely generated by a collection of operators"*.

The W-type $\mathrm{W}(A, B)$ is generated by

$$\mathsf{sup} : (a : A) \longrightarrow \big(B(a) \to \mathrm{W}(A, B)\big) \to \mathrm{W}(A, B)$$

where

- A is the type of *constructors*
- $B(a)$ gives the *arity* of constructor $a : A$

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Examples:

- For natural numbers $\mathbb{N}$, $A := 2$ and B is given by $B(\top) := 0$ and $B(\bot) := 1$.
- For lists $\mathsf{List}[C]$, $A := 1 + C$ and B is given by $B(\mathsf{inl}(-)) := 0$ and $B(\mathsf{inr}(-)) := 1$.

Well-founded trees

Principle of induction: *To prove that a property* $P(w)$ *holds for each well-founded tree* $w : \mathrm{W}$*, it suffices to prove that* $P(\sup(a, f))$ *holds whenever* $P(f\ b)$ *holds for each branch* $f(b) : \mathrm{W}$*.*

Type-theoretically: given a function

- $e : (a : A) \longrightarrow \big(f : B(a) \longrightarrow \mathrm{W}\big) \longrightarrow \big((b : B(a)) \longrightarrow P(f\ b)\big) \longrightarrow P(\sup(a, f))$

we have a function

- $\mathrm{F} : (w : \mathrm{W}) \longrightarrow P(w)$

such that

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Example: Defining $P(w) := \mathrm{W}$ and $e(a, -, g) := \sup(a, b \mapsto g(b))$ is an inductive way of defining the identity map on W.

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Let W be an initial algebra for the polynomial endofunctor

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- But: only if we can show $\pi_1(\mathsf{F}_\Sigma(w)) = w$ for each $w : \mathsf{W}$.

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- So is the map $w \mapsto \pi_1(\mathsf{F}_\Sigma(w))$ since we have

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- The above two maps are thus equal and we are done.

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Homotopy Type Theory

An extension of intensional type theory with concepts motivated by abstract homotopy theory.

- Consistent: we have interpretations into Quillen model categories (Awodey, Warren '09), groupoids (Hofmann, Streicher '96), simplicial sets (Voevodsky et al. '12), cubical sets (Bezem, Coquand, et al. '14).

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- Type-theoretic reasoning can lead to a novel proof of a known result, e.g., the fundamental group of the circle $\pi_1(\mathsf{S}^1)$ (Licata, Shulman '12).
- We can use geometric intuition to motivate further type-theoretic constructs.

The new homotopical interpretation

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We have the following interpretation into topological spaces:

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\begin{aligned}
\text{Types} &\quad\rightsquigarrow\quad \text{Spaces} \\
\text{Terms} &\quad\rightsquigarrow\quad \text{Points} \\
a : A &\quad\rightsquigarrow\quad \text{Point } a \in A \\
p : a =_A b &\quad\rightsquigarrow\quad \text{Path from } a \text{ to } b \text{ in } A \\
\alpha : p =_{(a =_A b)} q &\quad\rightsquigarrow\quad \text{Homotopy from } p \text{ to } q \text{ in } A \\
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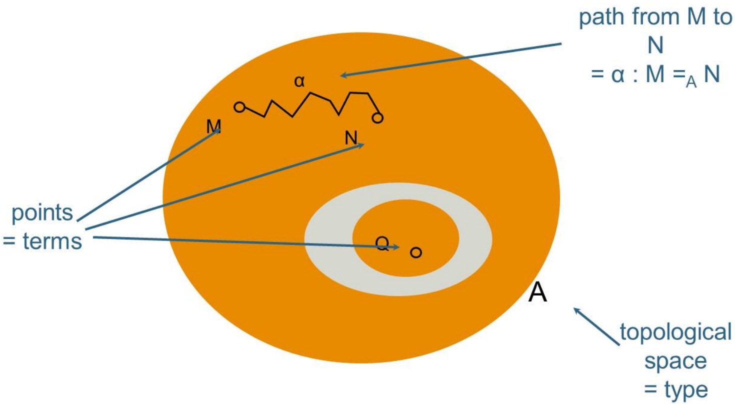
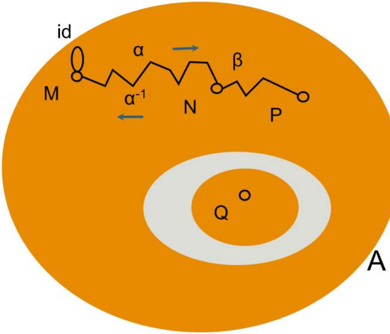
Types as Spaces



Operations on Paths



$id_M : M =_A M$

$\alpha^{-1} : N =_A M$

$\beta \circ \alpha : M =_A P$

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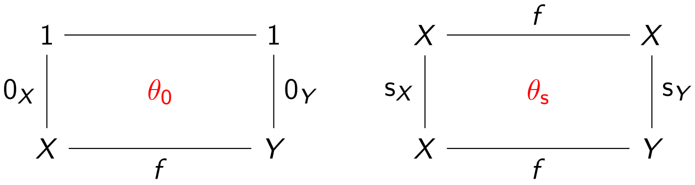
- A *morphism* $(X, 0_X, s_X) \longrightarrow (Y, 0_Y, s_Y)$ is a triple (f, θ_0, θ_s), where $f : X \longrightarrow Y$ and θ_0, θ_s witness the commutativity of the following two diagrams:



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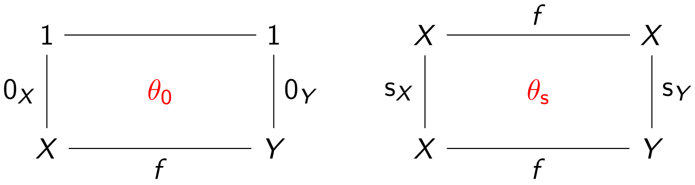

- An algebra $(\mathbb{N}, 0, \mathrm{suc})$ is *homotopy-initial* if the type of morphisms to any other algebra is *contractible*, *i.e.*, having a unique inhabitant up to equality.

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In homotopy type theory, we have a correspondence (Awodey, Gambino, S., '12) between

- Inductive types 0, 1, 2, $A + B$, $\mathbb{N}$, List[A], W(A, B) (with propositional computation rules)
- Homotopy-initial algebras for the appropriate endofunctors

So *e.g.*, $(\mathbb{N}, 0, \mathsf{suc})$ is homotopy-initial among algebras of the form $(X, 0_X, \mathsf{s}_X)$.

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- We put $\mathsf{F}(w) := \alpha(w) \#_P\, \pi_2(\mathsf{F}_\Sigma(w))$, where we use $\alpha(w)$ to transport $\pi_2(\mathsf{F}_\Sigma(w))$ from the fiber $P\big(\pi_1(\mathsf{F}_\Sigma(w))\big)$ to the fiber $P(w)$.

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$$\begin{aligned}\pi_1\big(\mathsf{F}_\Sigma(\mathsf{sup}(a, f))\big) &= \pi_1\big(e_\Sigma(a, b \mapsto \mathsf{F}_\Sigma(f\ b))\big)\\ &= \mathsf{sup}\ \big(a, b \mapsto \pi_1(\mathsf{F}_\Sigma(f\ b))\big)\end{aligned}$$

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- The above two maps are thus equal but we are not done.

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It remains to show that the homotopy $\alpha(w) : \pi_1(\mathsf{F}_\Sigma(w)) = w$ induced by the equality of the two morphisms $w \mapsto w$ and $w \mapsto \pi_1(\mathsf{F}(w))$ is *coherent*:

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- For any $a : A$ and $f : B(a) \longrightarrow \mathsf{W}$, the following diagram commutes:

$$\begin{array}{ccc} \pi_1\big(\mathsf{F}(\sup(a, f))\big) & \overset{\alpha(\sup(a,f))}{\rule{6em}{0.4pt}} & \sup(a, f) \\ \Big| & = & \Big| \\ \sup\big(a, b \mapsto \pi_1(\mathsf{F}_\Sigma(f\ b))\big) & \underset{\textit{via}\ \mathrm{funext}(b \mapsto \alpha(f\ b))}{\rule{6em}{0.4pt}} & \sup(a, b \mapsto f\ b) \end{array}$$

Conclusion

There is a similar (but much more complicated) correspondence between:

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Moreover, we know that:

- even more complicated higher inductive types such as set and groupoid quotients are special cases of W-quotients
- hence set and groupoid quotients inherit the characterization as homotopy-initial algebras