Opetopes, opetopic sets and polygraphs

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Thanks to

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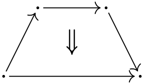
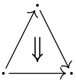
n-opetopes (for $n \leq 2$)

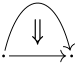
- There is a unique 0-dimensional opetope: the point (an operation with no input).

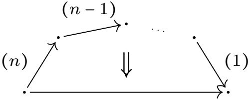
- There is a unique tree of 0-opetopes, yielding the unique arrow-shaped 1-opetope.

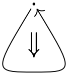
- 1-opetopes can assemble only as linear trees, and hence 2-opetopes are in one-to-one correspondence with natural numbers:

$\mathbf{3}$ = 

$\mathbf{2}$ = 

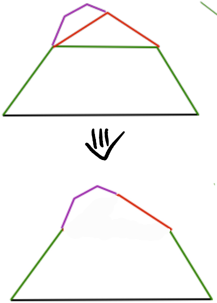
$\mathbf{1}$ = 

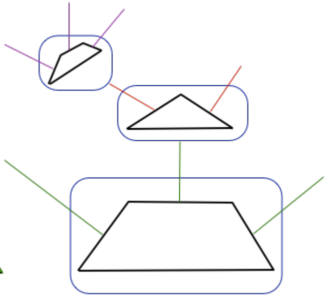
$\mathbf{n}$ = 

$\mathbf{0}$ = 

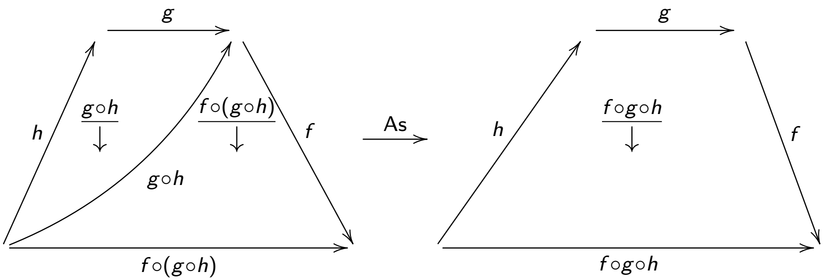
degenerate opetope

3-opetopes as trees





3-opetopes as unbiased associators



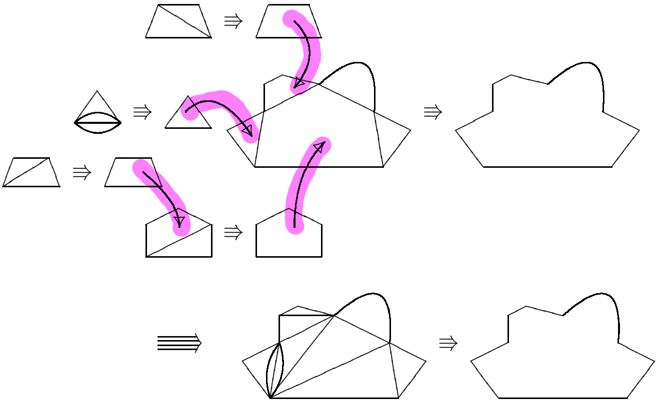
This picture features (decorated)

- 0-opetopes (unnamed)
- 1-opetopes ($f, g, h, g \circ h$,...)
- 2-opetopes (witnesses of unbiased composition $\underline{f \circ g \circ h}$,...)
- one 3-opetope (unbiased associativity)

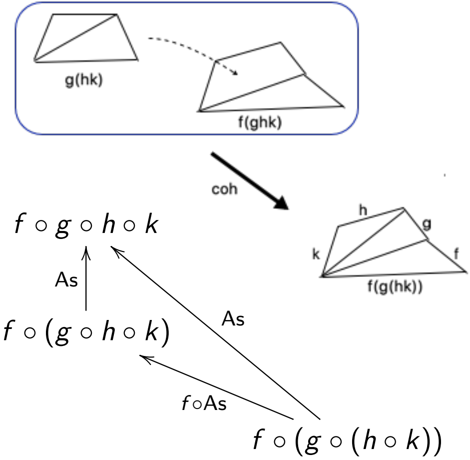
Contrast with the biased one: $f \circ (g \circ h) = (f \circ g) \circ h$

An example of 4-opetope

(taken from the beautiful Lauda-Cheng notes)

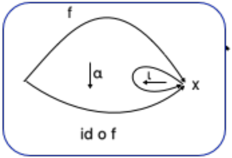


Unbiased coherence via 4-opetopes



5-opetopes, etc. feature higher coherences (trees of trees of...)

Identities via degenerate opetopes



Unit-left

This (poor) picture features

- the 2-opetope ι as a witness of the degeneracy promoting x to id_x
- the 2-opetope α as a witness of $\mathrm{id}_x \circ f$
- the 3-opetope Unit-left as the unit law $\mathrm{id}_x \circ f \to f$

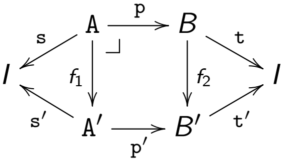
Note that ι has no sources (tree reduced to a leaf edge).

Polynomial functors (standard presentation)

Polynomial functors are triples of maps

$$I \xleftarrow{\mathtt{s}} \mathtt{A} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} J$$

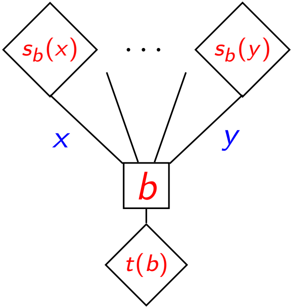
We are interested in polynomial endofunctors, i.e. $I = J$. A morphism of polynomial endofunctors is given by maps f_1, f_2 as below:



The pullback ensures that an operation b with arity $p^{-1}(b)$ is mapped to an operation with equipotent arity.

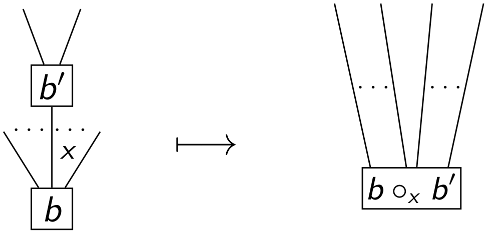
Polynomial functor (pictorially)

- We view B as a set of operations.
- For each operation b, we view $\mathtt{A}(b) = p^{-1}(b)$ as the arity of b.
- We view B as a set of colours, or of sorts (set of incoming edges).



Note the difference between names and decorations: the latter can be repeated, while the former are in bijection with the number of wires going into the operation.

Polynomial monad



Polynomial monads versus operads

Polynomial monads are a version of (set) operads that are

- Σ-free (the action of the symmetric group is free)
- non-skeletal (inputs are named, rather than numbered)
- described in the partial or "circle i" style
- coloured (or multisorted)

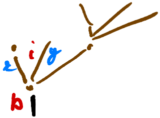
Note that the mechanics of polynomial functors dictates that the renaming of wires after composition be specified as part of the data defining the structure (cf. map f_1 above).

Polynomial monads are exactly the version of multicategories given by Hermida, Makkai and Power.

Free polynomial monad (trees)

Let P be a polynomial endofunctor on I. We define a new polynomial endofunctor $P^\star$ on I.
The operations are P-trees, i.e. trees with leaf edges where

- nodes are decorated by operations of P,
- incoming edges of a node decorated by b are in one-to-one correspondence with $\mathtt{A}(b)$,
- edges are decorated by colours of I



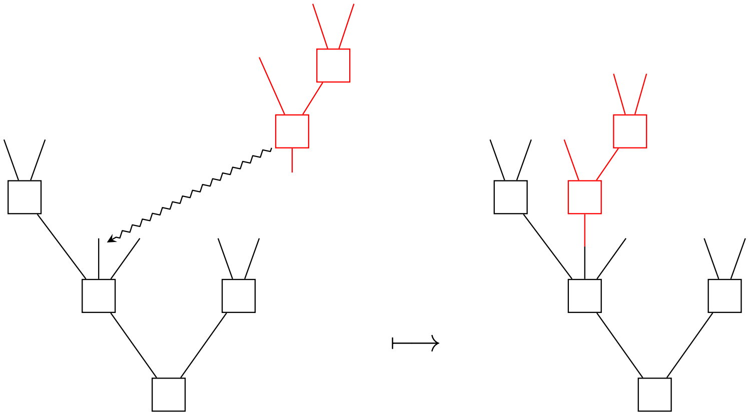
In $P^\star$:

- the arity of a tree T is the set of the occurrences of its **leaves**
- the target colour of T is the colour of the root of T

A P-tree may be reduced to a leaf (no node): we call it then degenerate.

Composition is defined by *grafting*.

The star multiplication (pictorially)



Another monad on trees: the + construction (Baez-Dolan)

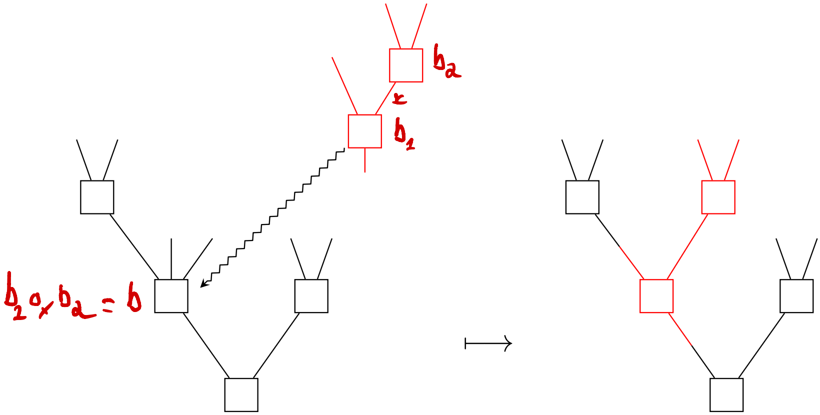
Here we follow Kock-Joyal-Batanin-Mascari 2010.

We now suppose that P is a polynomial monad on some I. Then the same P-trees give rise to another polynomial monad P^+, not on I, but on $B = B^P$:

- The arity of a tree is not its set of leaves anymore, but its set of **nodes**
- The target colour of T is $[\![T]\!]^{P^\star}$, where $[\![T]\!]$ is the evaluation of T according to the monad structure of P.
- Composition is by zooming in and *substituting* in nodes.

By iterating this construction, we shall get trees of trees of . . . !

The plus multiplication (pictorially)



Opetopes

Opetopes are defined by iteration of the + construction.

- Basis = identity polynomial functor $\mathcal{O}^0$on a singleton set

$$\{\blacklozenge\} \longleftarrow \{*\} \longrightarrow \{\blacksquare\} \longrightarrow \{\blacklozenge\}$$

There is only one 0-opetope $\blacklozenge$, and there is only one 1-opetope $\blacksquare$ which has only one input $*$, decorated by the unique 0-opetope $\blacklozenge$.

- Induction: We set

$$\mathcal{O}^n = (\mathcal{O}^{n-1})^+$$

and we write $\mathcal{O}^n$ as

$$\mathbb{O}_n \longleftarrow \mathbb{O}_{n+1}^\bullet \longrightarrow \mathbb{O}_{n+1} \longrightarrow \mathbb{O}_n$$

(the operations of $\mathcal{O}^{n-1}$ become the colours of $\mathcal{O}^n$)

A hierarchy of shapes

An n-opetope (for $n \geq 2$) is an oriented n-dimensional volume whose boundary is divided into a pasting scheme of source $(n-1)$-opetopes and a single target $(n-1)$-opetope.

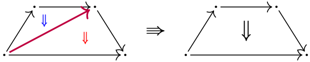
The target is determined by the pasting scheme of sources. Therefore, n-opetopes can be identified with pasting schemes of $(n-1)$-opetopes.

Pasting schemes of $(n-1)$-opetopes are described by trees whose nodes are decorated by $(n-1)$-opetopes and whose edges are decorated by $(n-2)$-opetopes.

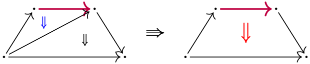
The category Ope

It has as objects all opetopes, and morphisms by generators s_x (for each node of the tree) and t, and relations

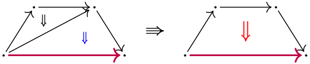
(Inner) $\quad \mathsf{s}_x\,\mathsf{s}_u = \mathsf{s}_y\,\mathsf{t} \quad$ (all edges)



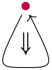
(Glob↑) $\quad \mathsf{t}\,\mathsf{s}_u = \mathsf{s}_x\,\mathsf{s}_u \quad$ (all leaves, ω non degenerate)



(Glob↓) $\quad \mathsf{s}_x\,\mathsf{t} = \mathsf{t}\,\mathsf{t} \quad$ ($x =$ root, ω non degenerate)



(Degen) $\quad \mathsf{t}\,\mathsf{s}_* = \mathsf{t}\,\mathsf{t} \quad$ (ω degenerate)



Opetopic sets are presheaves over Ope.

Polygraphs (a.k.a. computads)

A polygraph is (a presentation of) a strict ω-category (i.e. all truncations are strict n-categories). It is given by the following data:

- a set $\mathcal{P}_0$ of generating 0-cells,
- a set $\mathcal{P}_1$ of generating 1-cells, each coming with specified source and target in $\mathcal{P}_0$. This gives rise to a free strit 1-category $\mathcal{P}_1^*$ over these generators.

  $\vdots$

- a set $\mathcal{P}_{n+1}$ of $(n+1)$-generating cells, each coming with a specified source and target in $\mathcal{P}_n^*$. This gives rise to a free strict $(n+1)$-category $\mathcal{P}_{n+1}^*$ over these generators.

  $\vdots$

Polygraphic syntax

The n cells (or n-morphims) of $\mathcal{P}_n^*$ are equivalence classes of n-terms built via the following rules:

for $n=2$ $\circ_1$ = vertical
$\circ_0$ = horizontal comp.

- If $x \in \mathcal{P}_n$, then x is an $n-$term.
- If t is an $(n-1)$-term, then $\mathrm{id}(t)$ is an n-term.
- If t_1, t_2 are n-terms and $i < n$, then $t_1 \circ_i t_2$ is an n-term, provided $\mathfrak{s}^{n-i}s$ and $\mathfrak{t}^{n-i}t$ are provably equal as $(n-1)$-terms.

Sources and targets are derived information:

$$
\begin{array}{lll}
\mathfrak{s}\,(\mathrm{id}(t)) = t & \mathfrak{t}\,(\mathrm{id}(t)) = t & \\
\mathfrak{s}\,(t_1 \circ_i t_2) = \mathfrak{s}\,t_1 \circ_i \mathfrak{s}\,t_2 & \mathfrak{t}\,(t_1 \circ_i t_2) = \mathfrak{t}\,t_1 \circ_i \mathfrak{t}\,t_2 & (i < n-1) \\
\mathfrak{s}\,(t_1 \circ_{n-1} t_2) = \mathfrak{s}\,t_2 & \mathfrak{t}\,(t_1 \circ_{n-1} t_2) = \mathfrak{t}\,t_1 & (i < n-1).
\end{array}
$$

Equational theory (for n-terms)

$$
\begin{array}{ll}
(t_1 \circ_i t_2) \circ_i t_3 = t_1 \circ_i (t_2 \ \circ_i t_3) & \text{(category)} \\
\mathrm{id}^{n-i}(\mathfrak{s}^{n-i}t) \circ_i t = t \qquad t \circ_i \mathrm{id}^{n-i}(\mathfrak{t}^{n-i}t) = t & \text{(category)} \\
(s_1 \circ_i s_2) \circ_j (t_1 \circ_i t_2) = (s_1 \circ_j t_1) \circ_i (s_2 \circ_j t_2) \quad (i \neq j) & \text{(exchange law)} \\
\mathrm{id}(t_1) \circ_i \mathrm{id}(t_2) = \mathrm{id}(t_1 \circ_i t_2) \quad (i < n-1) & \text{(2-category)}
\end{array}
$$

Occurrences of generating n-cells in an n-cell

Let t be an n-term. We say that

- if $x \in \mathcal{P}_n$, then x has one occurrence (of a generating n-cell) (labelled by x),
- $\mathrm{id}(t)$ has no occurrence,
- the set of occurrences of $t_1 \circ_i t_2$ is the (disjoint) union of the sets of occurrences of t_1 and t_2.

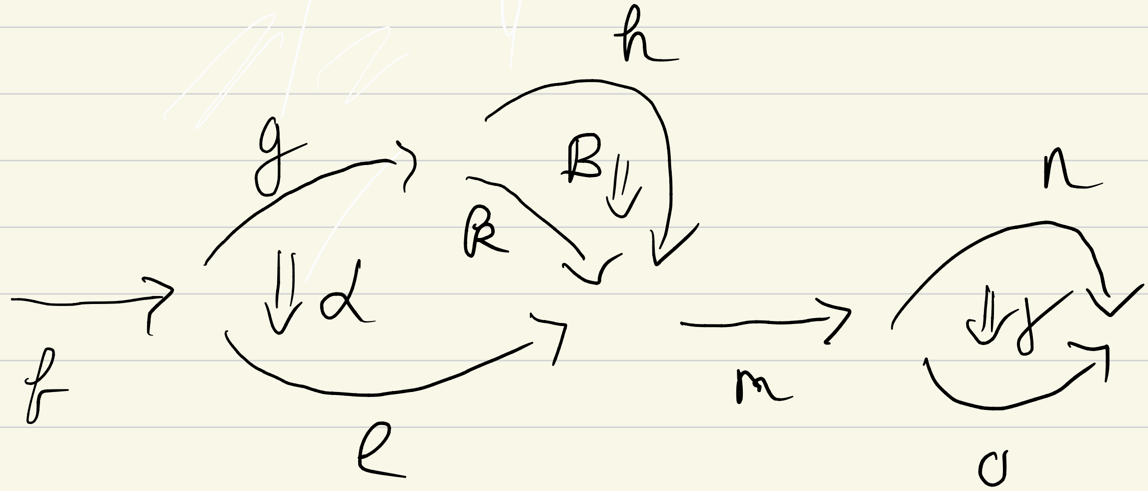
This notion is invariant under the choice of representatives of t.

It can be formulated using the language of contexts.

An *n*-context is a term with one occurrence of a special n-term $[]$, with specified source and target, called the hole.
We use the notation $C[]$ for the context, and $C[s]$ for the result of replacing $[]$ with some actual n-term s with the same source and target as the hole. This is called filling the hole.

Then occurrences (with their labelling generating n-cell) of t are in bijection with the pairs $(C[], x)$ such that $x \in \mathcal{P}_n$ and $t = C[x]$.

Illustrating occurrences



$$\left(\gamma \circ_0 id(m) \circ_0 \left[\alpha \circ_1 (B \circ_0 id(g)) \circ_0 id(f)\right]\right)$$

Occurrences α, B, γ

Many-to-one polygraphs

A polygraph is called many-to-one if for all n and $x \in \mathcal{P}_n$, we have $\mathfrak{t}\,x \in \mathcal{P}_{n-1}$ (all generating cells have as target a generating cell).

Theorem. Many-to-one polygraphs are the same thing as opetopic sets (giving rise to an equivalence of categories).

- The theorem has been proved by Victor Harnik, Michael Makkai and Marek Zawadowski (HMZ), replacing "opetopic" with "multitopic". (On the other hand, Eugenia Cheng has proved the equivalence between multitopic sets and opetopic sets.)
- Cédric Ho Thanh has a more direct proof, relying in part on notions and results of Simon Henry.
- Here, we offer a "plug-in" in Cédric's proof, making it entirely self-contained.

Remark. It follows from Henry's work that many-to-one polygraphs form a presheaf category $\mathrm{Set}^{(??)^{op}}$(without an explicit description of ??).

The key lemma

A many-to-one polygraph gives rise naturally to a family of polynomial endofunctors $\nabla_n\mathcal{P}$ (for $n \geq 1$):

$$\mathcal{P}_{n-1} \xleftarrow{\mathtt{s}} \mathtt{A}_n \xrightarrow{\mathtt{p}} \mathcal{P}_n \xrightarrow{\mathtt{t}} \mathcal{P}_{n-1}$$

where $\mathtt{A}_n(t)$ is the set of occurrences of $(n-1)$-generating cells of $\mathfrak{s}\, t$, and where $\mathtt{s}$ is the corresponding labelling (or filling).

Let $\mathcal{P}$ be a many-to-one polygraph. We write $\mathcal{P}_n^{mto}$ for the set of many-to-one n-cells, i.e. the cells whose target is a generating cell.

Lemma. For all n, there exists a bijective correspondence between $\mathcal{P}_n^{mto}$ and the set $\mathrm{Tr}\nabla_n\mathcal{P}$ of $(\nabla_n\mathcal{P})$-trees.

- There exists a composition map $(\_)^\circ : \mathrm{Tr}\nabla_n\mathcal{P} \to \mathcal{P}_n^{mto}$ based on a notion of placed composition defined by HMZ.
- Ho Thanh proves that $(\_)^\circ$ is bijective using some machinery developped by Henry.
- We provide here an **explicit inverse** to $(\_)^\circ$.

Down-to-earth reading of the key lemma

A $\nabla_n \mathfrak{P}$-tree has

- nodes decorated by generating n-cells
- edges decorated by generating $(n-1)$-cells

Looks like a (decorated) opetope!

Sketch of the proof of the theorem from the lemma

Here is the skeleton of the rest of Cédric's proof.

- One defines a realisation functor $|\_| : \mathrm{Ope} \to \mathcal{P}ol^{mto}$ (idea: name all sources and targets of an opetope). The goal is then to show that the induced adjunction

$$((\text{left Kan extension of } |\_|) \dashv \text{nerve})$$

  is actually an equivalence.
- The key lemma
  - allows to define a shape function from $\mathcal{P}_n$ to $\mathbb{O}_n$ (hereditarily use the key lemma, stripping the decorations by generating cells, and retaining only the underlying opetope),
  - and to establish a bijection

$$\text{between } \mathcal{P}_n \text{ and } \Sigma_{\omega \in \mathbb{O}_n} \mathcal{P}ol^{mto}(|\omega|, \mathcal{P}) = \Sigma_{\omega \in \mathbb{O}_n}(N\mathcal{P}_\omega)$$

    over $\mathbb{O}_n$ (restoring the decorations!).
  - This allows to prove that the unit and counit of the adjunction are isos.

Rest of the talk: proof of key lemma

(1) Recall the placed composition of HMZ and define the composition map $(\_)^\circ : \mathrm{Tr}\nabla_n\mathcal{P} \to \mathcal{P}_n^{mto}$. For this we need a tool/notation that we call context lifting.

(2) Define an invariant associated to evey cell (not only the many-to-one ones) = a forest, i.e. a (possibly empty) set of non-degenerate trees.

(3) Look more closely at this invariant when the cell is many-to-one: it provides the inverse of $(\_)^\circ$.

present contribution !

Lifting of contexts (in any polygraph)

If $C[\,]$ is an $(n-1)$-context (i.e. $(n-1)$-term with a hole), we define an n-context $C^{\uparrow}[\,]$ as follows:

- If $C[\,] = [\,]$, then we set $[\,]^{\uparrow} = [\,]$ where this n-hole can have any source s and target t such that $\mathfrak{s}\, s$ and $\mathfrak{s}\, t$ are both equal to the specified source of the original $(n-1)$-hole (and likewise for targets).
- If $C[\,] = t \circ_i C'[\,]$, then $C^{\uparrow}[\,] = \mathrm{id}(t) \circ_i C'^{\uparrow}[\,]$ (and symmetrically).

One can show that this definition is independent of the choice of representative of $C[\,]$.

We call $C^{\uparrow}[\,]$ the lifting of $C[\,]$.

Lifted contexts are whiskers!

$C[\,] = g \circ_0 [\,] \circ_0 f$ (1-context)

$C^{\uparrow}[\,] = \mathrm{id}(g) \circ_0 [\,] \circ_0 \mathrm{id}(f)$ (2-context)

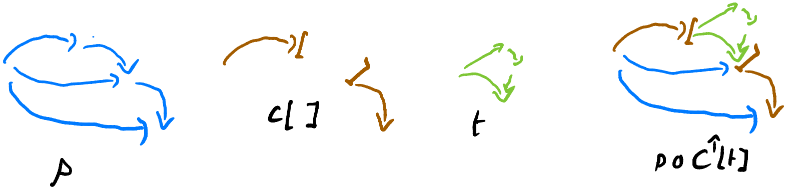
Placed composition (back to many-to-one polygraphs)

Consider two many-to-one cells s and t such that $\mathfrak{s}\, s = C[\mathfrak{t}\, t]$ for some context $C[\,]$. Then the term

$$s \circ C^{\uparrow}[t]$$

is well-defined and called the placed composition of s, t at $C[\,]$.

In dim. 2



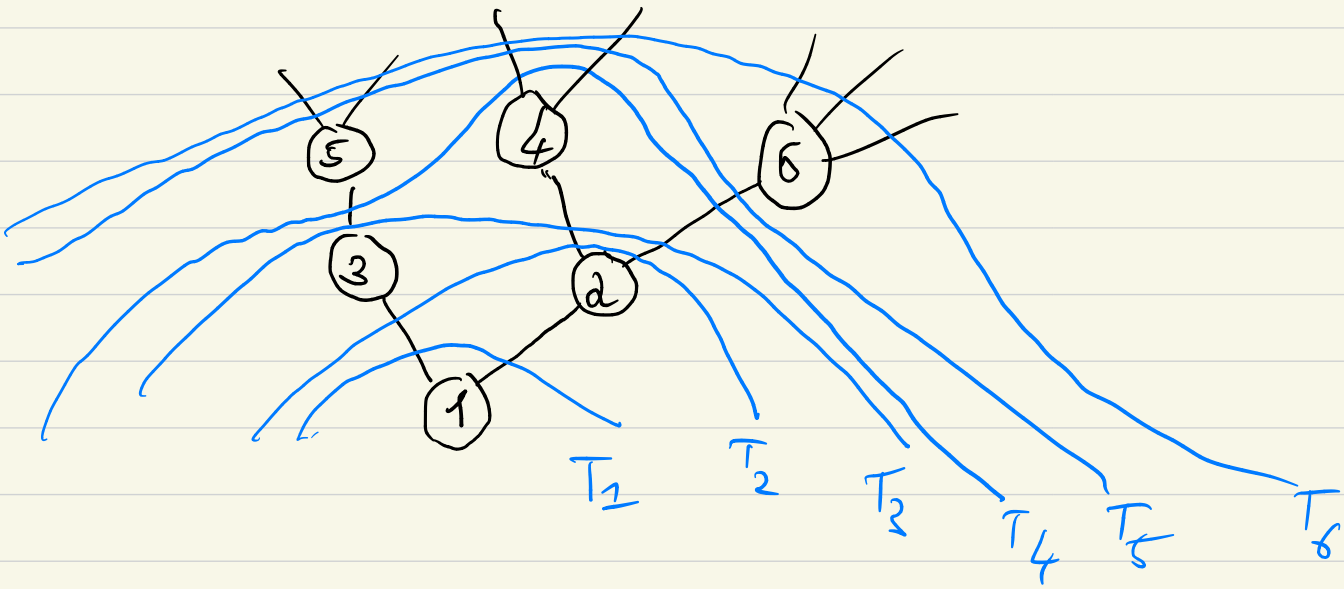
Composition of $(\nabla_n \mathcal{P})$-trees

- It T is degenerate, i.e., reduced to a leaf decorated with a $(n-1)$-generating cell y, then we set $T^\circ = \mathrm{id}(y)$.
- If T is non degenerate, i.e. has at least one node, we fix an admissible (i.e. ancestor respecting) enumeration of the nodes of T. This induces a sequence of trees: the i-th tree has the first i nodes of T, and the first one is just a single node tree decorated with generating cell x_1 (the root of T). We set $x_1^\circ = x_1$ and define T_{i+1}° as a placed composition of T° and x_{i+1} (the decoration of the $(i+1)$-th node) guided by the edge connecting the $(i+1)$-th node to T_i, which itself reads as a context by the definition of $(\nabla_n \mathcal{P})$.

That this definition is independent of the choice of admissible enumeration is a consequence of the following property, for $(n-1)$-contexts with two holes $C[\,]_1[\,]_2$ and generating n-cells x_1, x_2 such that $\mathfrak{t}\, x_i$ can fit in $[\,]_i$ $(i = 1, 2)$ (Godement rule!) :

$$C^{\uparrow_1}[x_1]_1[\mathfrak{t}\, x_2]_2 \circ_{n-1} C^{\uparrow_2}[\mathfrak{s}\, x_1]_1[x_2]_2 = C^{\uparrow_2}[\mathfrak{t}\, x_1]_1[x_2]_2 \circ_{n-1} C^{\uparrow_1}[x_1]_1[\mathfrak{s}\, x_2]_2$$

Illustrating admissible enumerations



The other way around: the forest of a cell (setting the scene)

We shall associate with every representative t of a cell in $\mathcal{P}_n^*$ a forest $\#(t)$ whose nodes are decorated by elements of $\mathcal{P}_n$ and whose edges are decorated by elements of $\mathcal{P}_{n-1}$, in such a way that the following properties hold.

- The set of leaf edges (resp. of root edges) of $\#(t)$ is in bijective correspondence with a subset L (resp. R) of nodes of the forest recursively associated with the source of t (resp. the target of t), and the bijection preserves the decorations.
- The set of nodes of $\#(\mathfrak{s}\, t)$ that are not in L is in bijective correspondence with the set of nodes of $\#(\mathfrak{t}\, t)$ that are not in R. We abuse notation by writing this as

$$\#(\mathfrak{s}\, t) \setminus \text{leaves}(\#(t)) = \#(\mathfrak{t}\, t) \setminus \text{roots}(\#(t)).$$

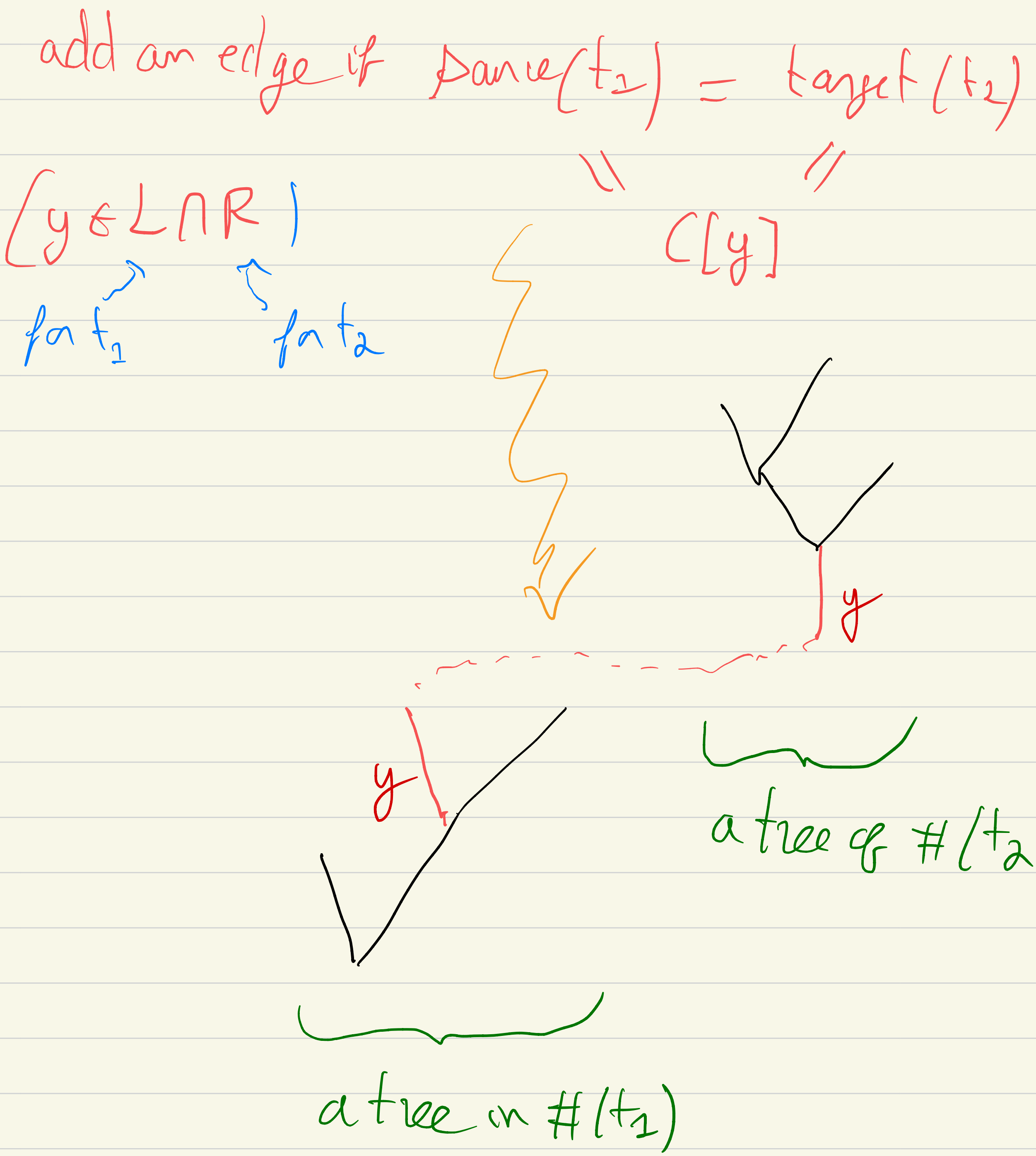
The forest of a cell (definition)

(the polygraph is many-to-one, the cell is arbitrary)

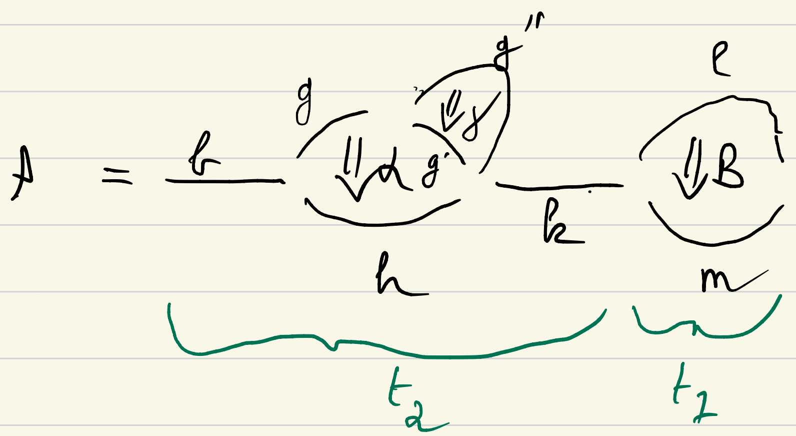
- If $t = x$ is a generating n-cell, then $\#(t)$ is forest consisting of one tree reduced to one node, decorated by x. The leaf edges of the forest are in one-to-one correspondence with the nodes of $\#(\mathfrak{s}\, x)$ and receive the corresponding decorations, and the root edge is decorated with $\mathfrak{t}\, x$.
- If $t = \mathrm{id}(t')$, then we set $\#(t)$ to be the empty forest (whatever t' is).
- If $t = t_1 \circ_i t_2$, with $i < n - 1$, then $\#(t)$ is the disjoint union of the forests $\#(t_1)$ and $\#(t_2)$.
- If $t = t_1 \circ_{n-1} t_2$, then $\#(t)$ is obtained by grafting some trees of $\#(t_2)$ above $\#(t_1)$: if a root u of $\#(t_2)$ is such that $u \in L$ (L relative to t_1), we graft the tree of root u of $\#(t_2)$ on the corresponding tree of $\#(t_1)$.

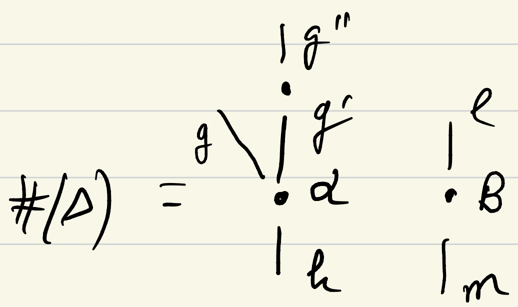
This definition does not depend on the choice of a representative of an n-cell.

Generic illustration for $\#(t_1 \circ_{n-2} t_2)$



Generic illustration for $\#(t_1 \circ_i t_2)$ $(i < n-1)$





$\#(\mathrm{Dance}(A))^\bullet = \{b, g, g'', k, \ell\}$ $\quad L = \{g, g'', \ell\}$

$\#(\mathrm{target}(A))^\bullet = \{b, h, k, m\}$ $\quad R = \{h, m\}$

$\#(\mathrm{Dance}(A))^\bullet \setminus L = \{b, k\} = \#(\mathrm{target}(A))^\bullet \setminus R$

Remark The discrepancy between leaves $(\#(A))$ and nodes of $\#(\mathrm{Dance}(A))$ arises only from the compositions $-\circ_i-$ $\quad i < n-1$ [for p in P_n^*].

Canonical forms for many-to-one cells

Proposition. Any many-to-one n-cell has a representative t that has one of the following shapes:

- $t = x$ for $x \in \mathcal{P}_n$,
- $t = \mathrm{id}(y)$ for $y \in \mathcal{P}_{n-1}$,
- $t = t_1 \circ_{n-1} t_2 \ldots \circ_{n-1} t_n$ $(n \geq 2)$, where
  - $t_1 = x_1 \in \mathcal{P}_n$, and
  - each t_i $(i > 1)$ is of the form $C_i^{\uparrow}[x_i]$, where $x_i \in \mathcal{P}_n$ and $\mathfrak{s}\, t_{i-1} = C[\mathfrak{t}\, x_i]$.

In plain words, t is a placed composition of the generating n-cells occurring in it.

The tree associated with a many to one cell

Corollary. If t is many-to-one, then we have

- $\#(t)$ is empty $\Leftrightarrow t = \mathrm{id}(y)$ for some $y \in \mathcal{P}_{n-1}$.
- $\#(t)$ is not-empty $\Leftrightarrow \#(t)$ consists of a single (non-degenerate) tree. Moreover the set of leaves of $\#(t)$ is in one-to-one correspondence with the set of nodes of $\#(\mathfrak{s}\, t)$ (i.e. $\#(\mathfrak{s}\, t) \setminus L = \emptyset$).

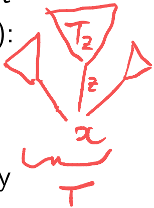
This allows us to define $\underline{\#} : \mathcal{P}_n^{mto} \to \mathrm{Tr}\nabla_n\mathcal{P}$ by

- $\underline{\#}(\mathrm{id}(y))$ is the degenerate tree whose unique leaf is decorated with y,
- $\underline{\#}(t) = \#(t)$ otherwise.

Morale. Even in the canonical forms of many-to-one cells, t_i for $(i > 1)$ is not many-to-one in general. This is why we had to define a wider invariant (forests) working for all cells, and only then narrow it down to the many-to-one cells.

$(\_)^\circ$ and $\underline{\#}$ are inverse

- $(\_)^\circ \circ \underline{\#} = \mathrm{id}$. Clear for $t = x$.
  - If $t = \mathrm{id}(y)$, then $\underline{\#}(x)$ is the degenerate tree decorated with y, hence $(\underline{\#}(x)) = \mathrm{id}(y) = t$.
  - If $t = x_1 \circ_{n-1} t_2 \ldots \circ_{n-1} t_n$, the inductive definition of $\#(t)$ provides an admissible enumeration for $\#(t)$, composing along which yields exactly the same representative t we started from.
- $\underline{\#} \circ (\_)^\circ = \mathrm{id}$. Clear for degenerate T. If $T = x\{z \leftarrow T_z \mid z \in Z\}$, then we fix an order $Z = \{z_1 < \cdots < z_p\}$, take adm. enum. on each T_{z_i}: this determines an adm. enum. of T. One shows that composing along it gives (for suitable lifted contexts $C_i^\uparrow[]$):



$$T^\circ = x \circ_{n-1} C_1^\uparrow[T_{z_1}^\circ] \circ_{n-1} \ldots \circ_{n-1} C_p^\uparrow[T_{z_p}^\circ]$$
$$\#(T^\circ) = x\{z_i \leftarrow \#(C_i^\uparrow[T_{z_i}^\circ]) \mid i = 1, \ldots, p\}$$

and we conclude by induction, thanks to the following easy

Lemma. If $C^\uparrow[]$ is a lifted context, then

$$\#(C^\uparrow[t]) = \#(t) \quad \text{(for all } t \text{ fitting in the hole)}$$

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