Polynomial Functors and Opetopic Types in Type Theory

2022 Workshop on Polynomial Functors

- Remarks

  ~ Type Theory is a proof relevant system

$$x : X \qquad (x \in X)$$

term — type

proofs — statements

$$n : \mathbb{N} \qquad + : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

I know a natural

data $\mathbb{N}$: Type
- $0 : \mathbb{N}$
- $S : \mathbb{N} \to \mathbb{N}$

$\overline{|SSS0|}$

~ We would a language rich enough to capture all mathematics ⇝ need notion of equality

$$\frac{x : X \qquad y : X}{x = y : \text{Type}}$$

proofs that x and y are equal.

$$\frac{p : x = y \qquad q : x = y}{p = q : \text{Type}} \quad \cdots\cdots$$

- Awodey-Warren, Voevodsky

  ⇝ Types are Homotopy types/∞-groupoids

- Suggests Type Theory is/should be a language for higher dim'l / homotopy th. reasoning.

Monoid in Type Theory

$X : \text{Type}$

$m : X \times X \to X$

$e : X$

$\to$ assoc : $m(x, m(y,z)) = m(m(x,y), z)$

assoc^2 :

$\vdots$

$\to$ Polynomial Monad? $\leftarrow$ * with/out extra assumptions

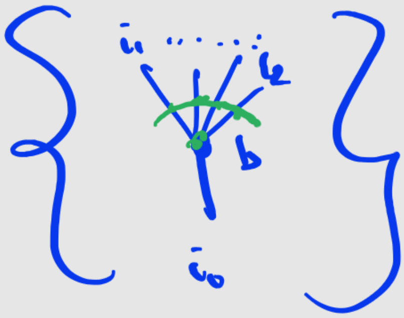
$\to$ Opetopic Approach to coherent structures.

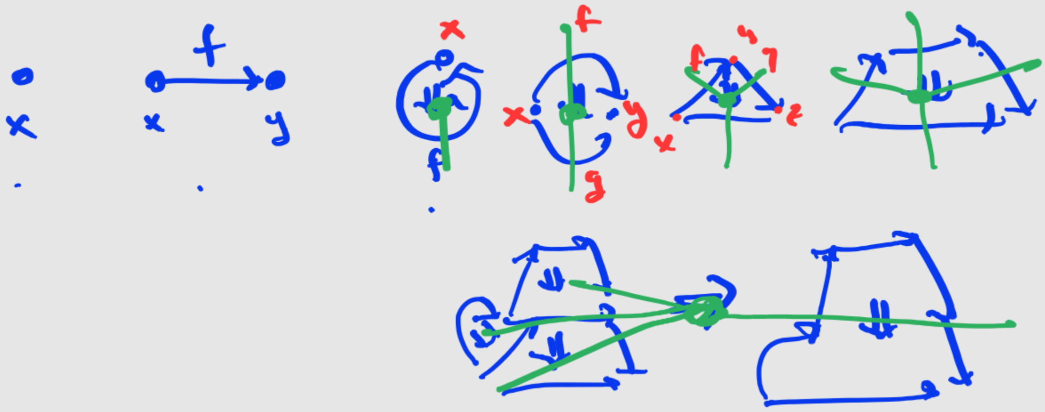
$$I \xleftarrow{s} E \to B \to I$$

$I : \text{Type}$

$B : I \to \text{Type}$

- $E : (i:I)(b:B\,i) \to \text{Type}$
- $s : (i:I)(b:B\,i)(e:E\,i\,b) \to I$



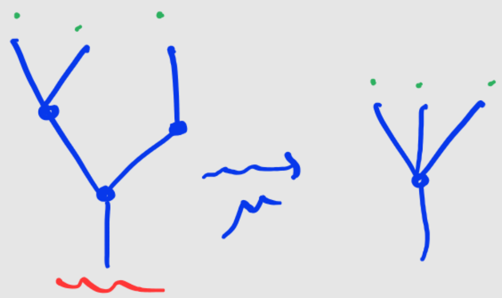


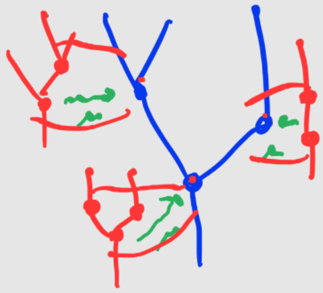
$P \quad I \leftarrow E \rightarrow B \rightarrow I$

$\rightarrow \mu : \underbrace{[P]B}_{} \xrightarrow[I]{} B$ + cartesian

$\eta : I \xrightarrow[I]{} B$

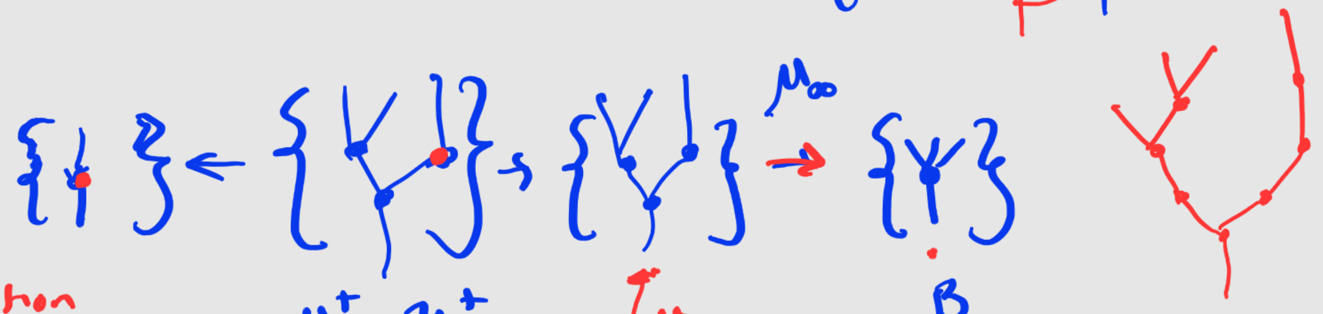
+ axioms





Baez-Dolan + constr.

μ^+
η^+



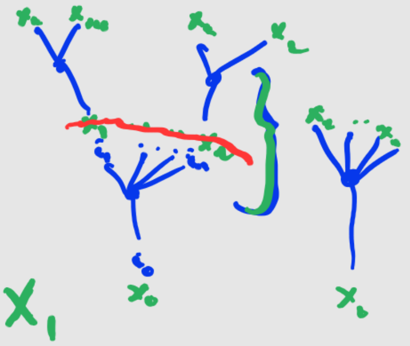
Free resolution of an R-module

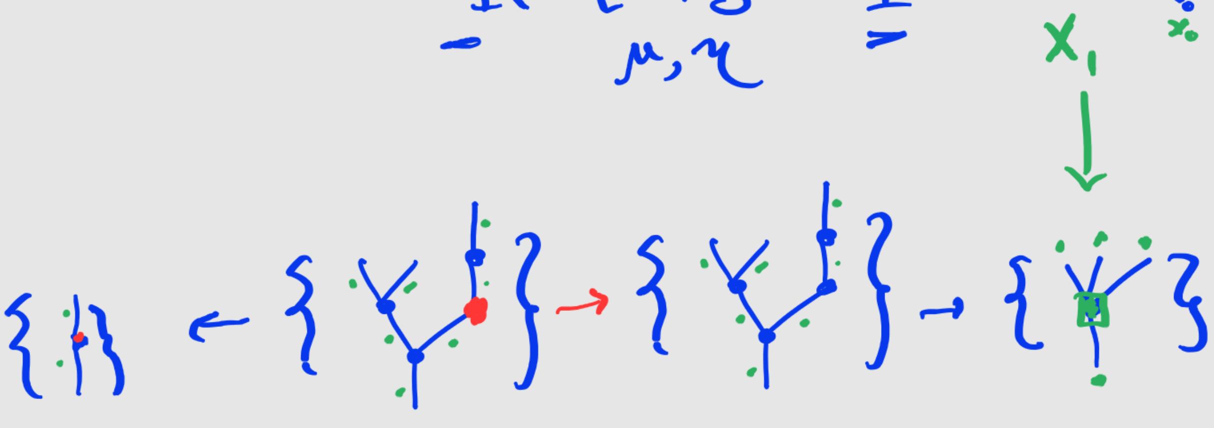
Consequence: $M \quad M^{+} \quad M^{++} \quad M^{+++} \quad \dashv \ \cdots$

An M-Opetopic
"The set underlying a coherent M-algebra"

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & E_X \arrow[r] \arrow[d, "\sim"] & B_X \arrow[r] \arrow[d] & X_0 \arrow[d] \\
\underline{I} & E \arrow[l] \arrow[r] & B \arrow[r] & \underline{I}
\end{tikzcd}$$

μ, η





An opetopic type is an infinite sequence

$$(X_0, X_1, X_2, \ldots\ldots)$$

$$\begin{tikzcd}
I & E \arrow[l] \arrow[r] & B \arrow[r] & I \\
I^+ & E^+ \arrow[l] \arrow[r] & B^+ \arrow[r] & I^+
\arrow[from=1-2, to=2-2, squiggly]
\end{tikzcd}$$

$\underline{\mu}, \underline{\eta}$

uses μ^+, η^+

$\mu(\mu \cancel{x})$

primitives of type

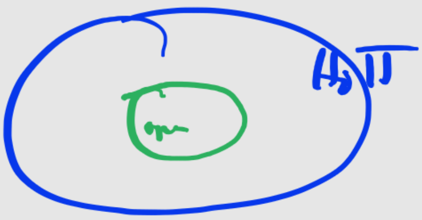
B^+	$\boxed{\mu}, \boxed{\eta}$
μ^+, η^+	assoc unit
assoc^+ unit^+	assoc^2 unita^2
$\vdots$	

Solution !? → Make μ, η primitive.

$$\left.\begin{cases} \bullet\ X \xrightarrow[o]{} Y \ \checkmark \\ \bullet\ \text{Dependent Opetopic Type} \ \checkmark \\ \bullet\ \Sigma, \Pi \end{cases}\right\} \ \underline{\underline{\text{Agda}}}$$

→ Compare

- HTS, 2LTT



- Opetopic

What is a HIT??

" $\mathcal{U}$ - opetopic "
type

- Definitions

→ Id

- ∞-groupoids
- $(\infty,1)$-categories
- (∞,n)-categories
- ∞-planar-operads + algebras
- A_∞-groups/monoids

→ Fin

Symmetric monoidal (∞,n)-categories

E_∞-groups monoids

→ Type

Poly Monad

M $\quad O(1) = \mathrm{Poly}$

Constructions

- $X \rightsquigarrow \mathrm{Id}_0(X)$
- $\mathcal{U}_{oR}$
- Free ∞-groupoid
- Join

$\longleftarrow \mathrm{Fin}^* \longrightarrow \mathrm{Fin} \longrightarrow \bullet$

Thm

$\mathrm{Type} \simeq \infty\text{-grp}$

$A_\infty\text{-grps} \simeq \mathrm{Ptd}\text{-Conn.}$

Type - $(\infty,1)$-category.

$\circ \longleftarrow \mathrm{Type}^{d} \longrightarrow \mathrm{Type} \longrightarrow *$

$\{i, E, E \to I\}$

$$\begin{tikzcd} & & & I \arrow[d] \\ 0 & U^{+} \arrow[l] \arrow[r] & U \arrow[r] & \bullet \end{tikzcd}$$