Familial Monads for Higher and Lower Category Theory

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Categories with Different Cell Shapes

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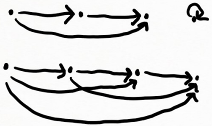

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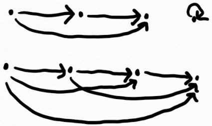

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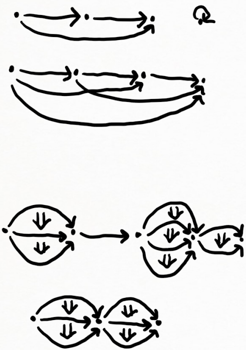

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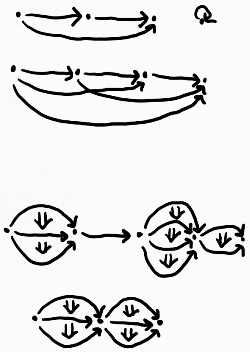

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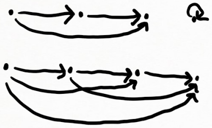

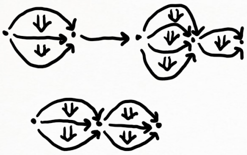

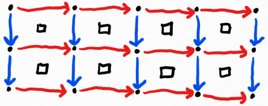

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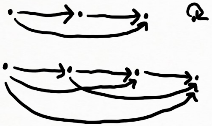

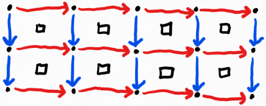

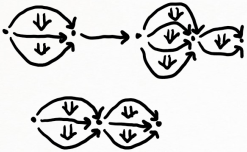

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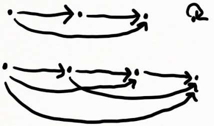

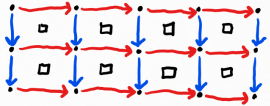

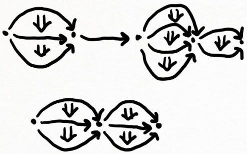

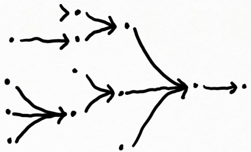

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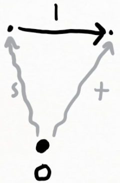

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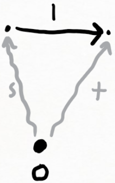

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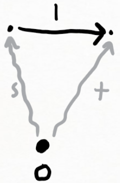

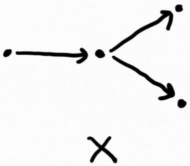

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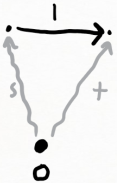

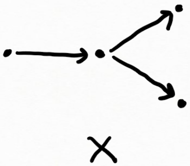

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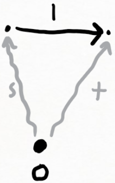

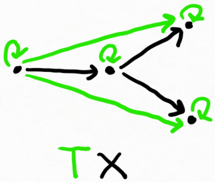

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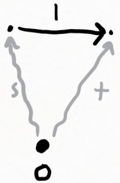

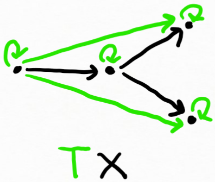

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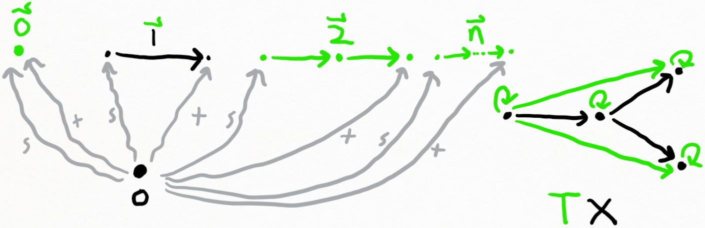

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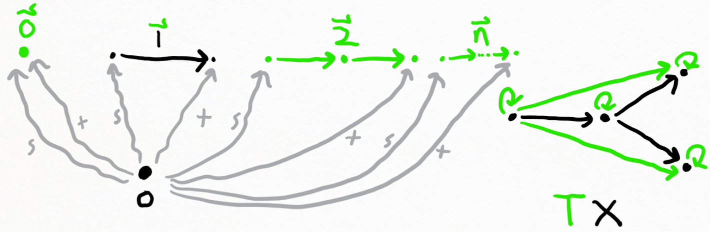

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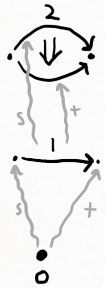

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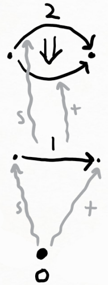

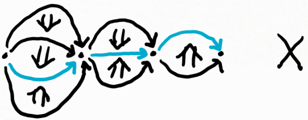

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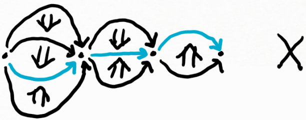

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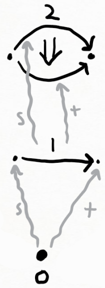

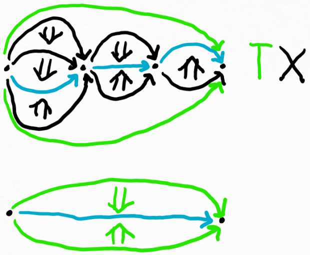

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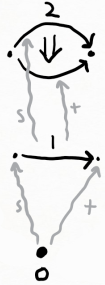

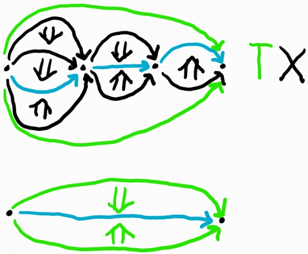

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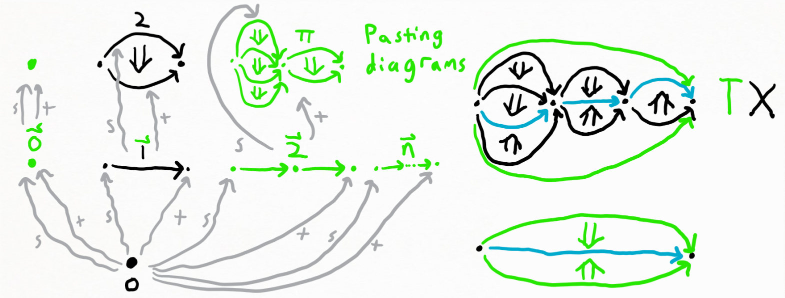

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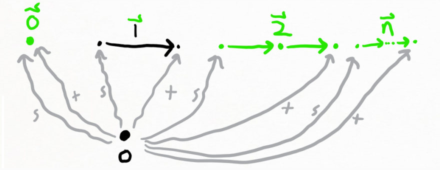

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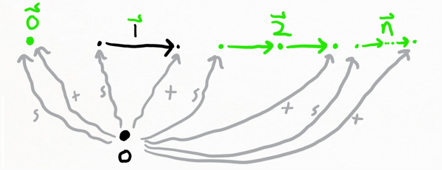

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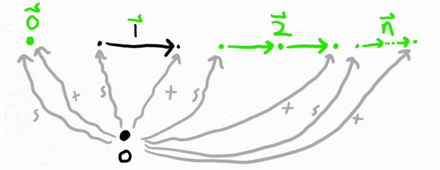

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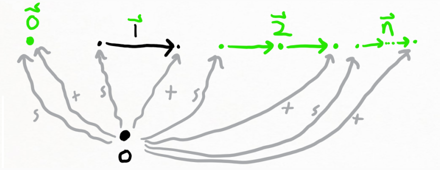

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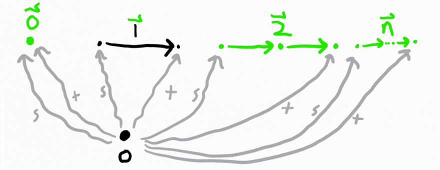

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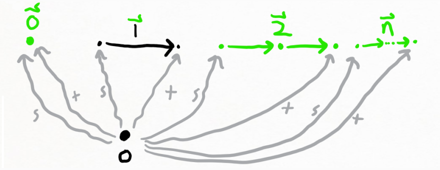

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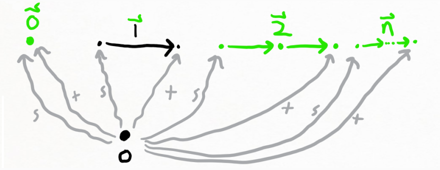

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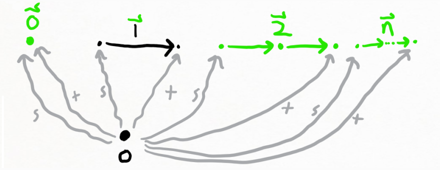

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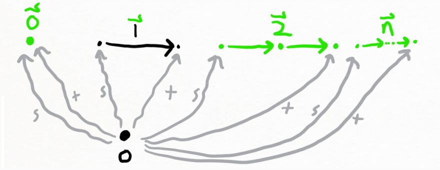

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- In this sense, algebraic higher categories "live in" *Poly*

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- They can model commutativity *up to a higher cell*, like in symmetric monoidal categories

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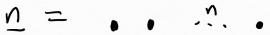

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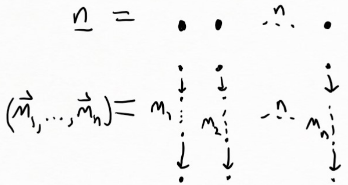

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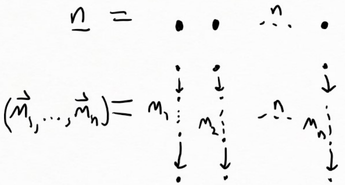

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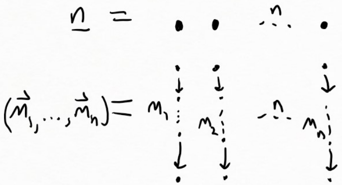

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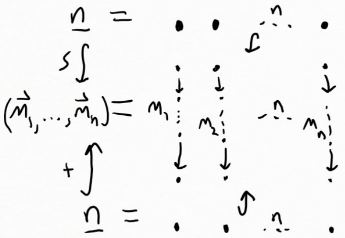

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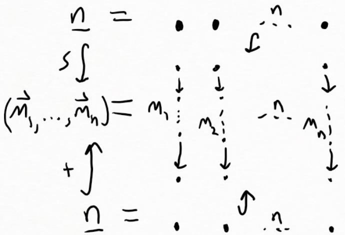

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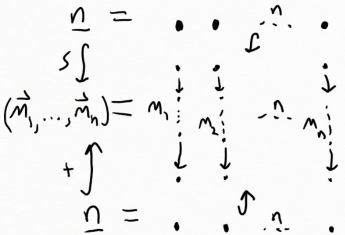

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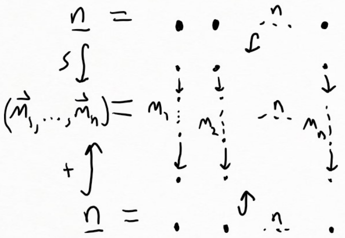

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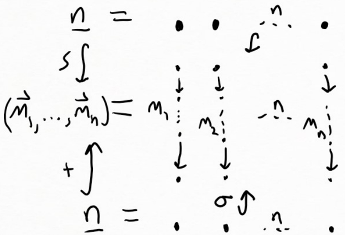

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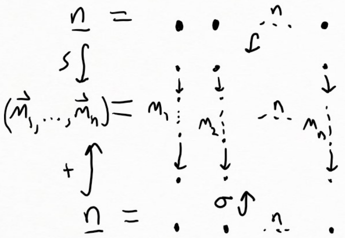

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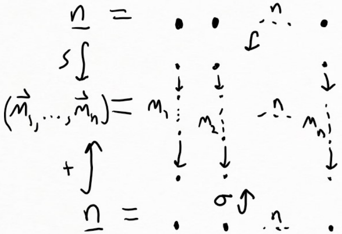

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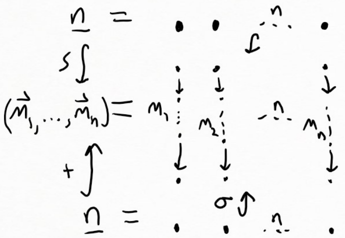

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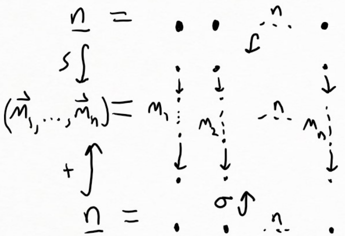

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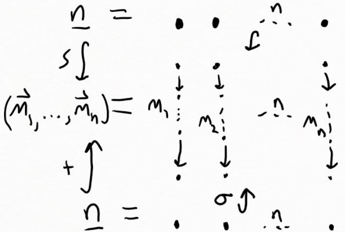
- The case $m_1 = \cdots = m_n = 1$ encodes naturality of the symmetries, and the monad structure ensures invertibility etc.



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- This diagram may be more easily described in *Poly* than the category of commutative monoids

References

- Brandon Shapiro, "Familial Monads as Higher Category Theories." arXiv:2111.14796
- David Spivak, "Functorial Aggregation." arXiv:2111.10968

Thanks!