Eilenberg MacLane polynomial functors

Eilenberg MacLane (1954)

G an abelian gp

$K(G,n)$ is a topological space

s.t. $\pi_i(K(G,\underline{\underline{n}})) = \begin{cases} G & \text{if } i=n \\ 0 & \text{otherwise.} \end{cases}$

$$\left.\begin{array}{ccc} Ab & \longrightarrow & Ab \\ G & \longmapsto & \underbrace{H_p}_{\text{singular homology}}(K(G,n),\mathbb{Z}) \end{array}\right\}$$ is a functor which is $\underline{\text{NOT}}$ additive.

EMcL polynomial functors

→ as a generalization of additive functors

→ a way to measure the complexity of a functor

→ as a tool to study non-additive functors.

<u>Plan</u>: I Definition and first pties

II Pties and equivalence def.

III Classification results

Introductory examples Ab: category of abelian group

- $A \in Ab$. $F_A : Ab \to Ab$.

$$F_A(G) = G \otimes A$$

$$F_A(G_1 \oplus G_2) = (G_1 \oplus G_2) \otimes A = F_A(G_1) \oplus F_A(G_2)$$

$\iff$ F_A is an additive functor

- $T^2 : Ab \to Ab$ $\quad T^2(G) = G \otimes G$

$$T^2(G_1 \oplus G_2) = T^2(G_1) \oplus \underbrace{G_1 \otimes G_2 \oplus G_2 \otimes G_1}_{\text{the defect of additivity}} \oplus T^2(G_2)$$

is not additive

second cross-effect $\leftarrow$ $cr_2 T^2 : Ab \times Ab \longrightarrow Ab$.

$$cr_2 T^2(G_1, G_2) = G_1 \otimes G_2 \oplus G_2 \otimes G_1$$

symmetric in the two variables

we can fix the second variable

$$cr_2 T^2(-, G_2) : Ab \longrightarrow Ab$$

$$cr_2 T^2(-, G_2) = \underbrace{- \otimes G_2}_{F_{G_2}} \oplus G_2 \otimes - \neq 0$$

$$= F_{G_2} \oplus \widetilde{F}_{G_2}$$

$$\underbrace{cr_3}_{3^{rd}\text{-cross-effect.}} T^2(-,-,G_2) := cr_2\big(cr_2 T^2(-, G_2)\big) = 0$$

$F : Ab \to Ab \rightsquigarrow cr_3 F : Ab \times Ab \times Ab \to Ab$

F poly of $\stackrel{o}{=}$ n <u>if</u> $cr_{n+1} F = 0$ and $cr_n F \neq 0$

<u>ex:</u> T^2 is poly of $\stackrel{o}{=}$ 2.

an additive functor is poly of $\stackrel{o}{=}$ 1.

The setting

$$F : \mathcal{C} \longrightarrow k\text{-Mod} \qquad k \text{ commutative ring.}$$

$(\mathcal{C}, \odot, 0)$ symm monoidal category

~~Sets~~

(or more generally an abelian category)

where the unit 0 is the null / zero object of $\mathcal{C}$

initial $\exists!\ 0 \to X$

& terminal $\exists!\ X \to 0$

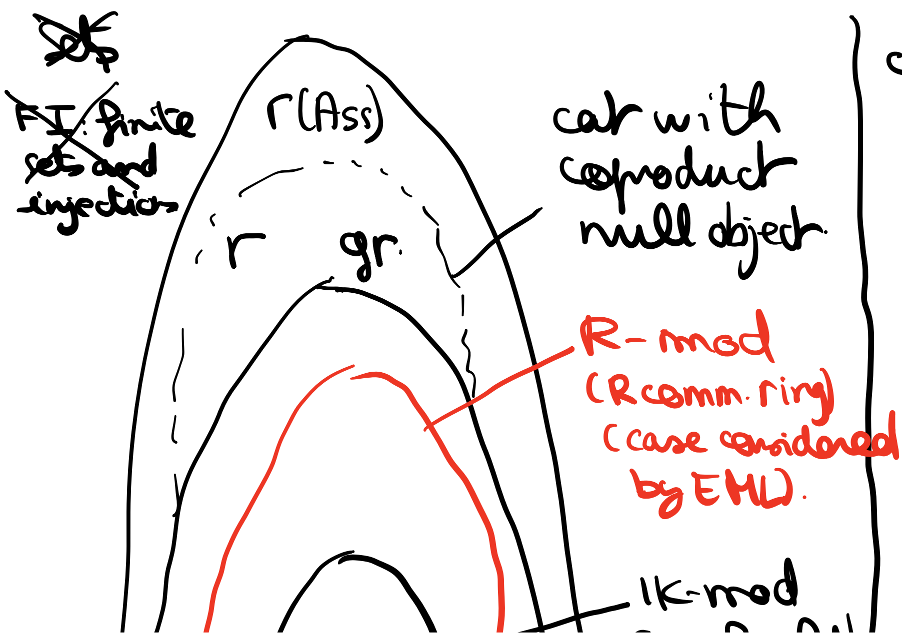
(for Gr: Baues, Baues-Pirashvili)

+ $\mathcal{C}$ small in order to consider

$\mathcal{F}(\mathcal{C}, k)$: category of functors $\mathcal{C} \to k\text{-Mod}$.

(it is an abelian category. we have ses of functors $0 \to F \to G \to H \to 0$)

Examples and non-examples of such category $\mathcal{C}$



$\mathcal{A}$ an additive category

$\mathcal{A}$ has a null object

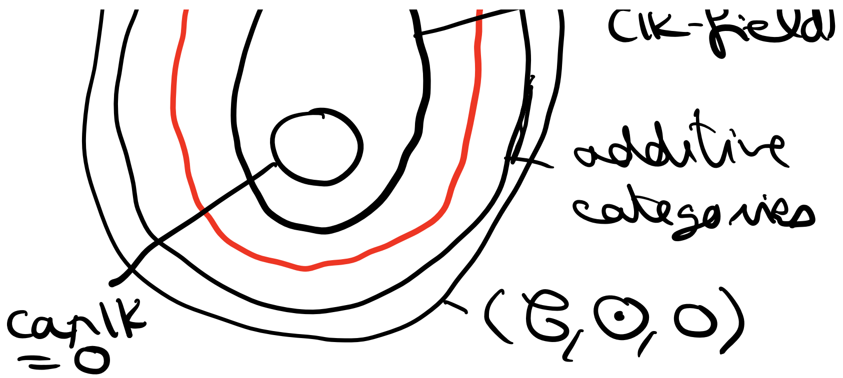
the biproduct $\oplus$ give rise to a symm monoidal cat $(\mathcal{A}, \oplus, 0)$

sets and partial map.

Γ : category of finite pointed sets

$[n] = \{\underline{0}, 1, \dots n\}$

pointed maps. $[n] \to [m]$, $0 \mapsto 0$



$[n] \sqcup [m] = [n+m]$ (coproduct which is *not* a product)

$[0]$ null object.

poly functors on Γ: Pirashvili in 2000

gr: category of finitely generated free groups

$F_n = \langle x_1, \dots, x_n \rangle$.

$F_n * F_m = F_{n+m}$ (coproduct = free product which is not a product)

0 null object.

poly functors on gr: Baues-Pirashvili (≤ 2) (1999)
Hartl-Pirashvili-V. (2015)
Djament-V. (2015)
Powell (2022)

R-mod: category of finitely generated free left R-modules

- $X \in \mathcal{C}$ $\qquad X \simeq X \odot 0 \underset{\text{sym}}{\simeq} 0 \odot X$

$$X_1 \odot \cdots \odot X_k \odot \cdots \odot X_n \xrightarrow[\text{(0 terminal)}]{r_k^n} X_1 \odot \cdots \odot 0 \odot \cdots \odot X_n$$

$$\simeq X_1 \odot \cdots \odot X_k \odot \cdots \odot X_n$$

$$X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n \simeq X_1 \odot \cdots \odot 0 \odot \cdots \odot X_n \xrightarrow[(0 \text{ initial})]{i_k^n} X_1 \odot \cdots \odot X_k \odot \cdots \odot X_n$$

$$r_k^n \circ i_k^n = \mathrm{Id}_{X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n}$$

$$F : \mathcal{C} \to k\text{-Mod}$$

__Def__: [cross-effects of F]

$$F(X_1 \odot \cdots \odot X_n) \xrightarrow{\begin{pmatrix} F(r_1^n) \\ \vdots \\ F(r_n^n) \end{pmatrix}} \bigoplus_{k=1}^{n} F(X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n)$$

$$cr_n F(X_1, \dots, X_n) := \ker \begin{pmatrix} F(r_1^n) \\ \cdots \\ F(r_n^n) \end{pmatrix}$$

$$cr_n F : \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n} \to k\text{-Mod}$$

$$\Longleftrightarrow \quad cr_0 F = F(0)$$

$cr_1 F = \mathrm{Ker}(F \longrightarrow F(0))$ $\quad F(i_2)$

$cr_2 F(X_1, X_2) = \mathrm{Ker}(F(X_1 \odot X_2) \longrightarrow F(X_2) \oplus F(X_1))$

for $n \geqslant 3$.

$$cr_n F(X_1, \dots X_n) := cr_2\big(cr_{n-1} F(-, X_3, \dots X_n)\big)(X_1, X_2)$$

Def: $F : \mathcal{C} \to k\text{-Mod}$ is poly of deg d.

if $cr_d F \neq$ and $cr_{d+1} F = 0$

$Pol_d(\mathcal{C}, k)$ the full subcategory of $\mathcal{F}(\mathcal{C}, k)$ of poly functors of deg $\leq d$

$$\cdots \hookrightarrow Pol_{d-1}(\mathcal{C}, k) \hookrightarrow Pol_d(\mathcal{C}, k) \hookrightarrow \cdots \hookrightarrow \mathcal{F}(\mathcal{C}, k)$$

Prop: $F : \mathcal{C} \to k\text{-Mod}$ st $F(0) = 0$

$$F(X_1 \odot \cdots \odot X_n) = \bigoplus_{k=1}^{n} \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} cr_k F(X_{i_1}, \dots, X_{i_k})$$

ex:

$$F(X_1 \odot X_2) = F(X_1) \oplus F(X_2) \oplus cr_2 F(X_1, X_2)$$

$$F(X_1 \odot X_2 \odot X_3) = F(X_1) \oplus F(X_2) \oplus F(X_2)$$
$$\oplus\, cr_2 F(X_1, X_2) \oplus cr_2 F(X_1, X_3)$$
$$\oplus\, cr_2 F(X_2, X_3)$$
$$\oplus\, cr_3 F(X_1, X_2, X_3)$$

Ex: $T^2 : Ab \to Ab$

$$\underline{T^2(G_1 \oplus G_2 \oplus G_3)} = (G_1 \oplus G_2 \oplus G_3) \otimes (G_1 \oplus G_2 \oplus G_3)$$
$$= \ldots$$

$$cr_3 T^2(G_1, G_2, G_3) = 0$$

$\Rightarrow$ T^2 is poly of ≤ 2.

Ex: $a : gr \to Ab$

$$G \mapsto a(G) = G/[G,G]$$

$$a(G * H) = a(G) \oplus a(H).$$

$\sigma_2 \alpha(G, N) = 0$

α is poly of $\stackrel{\circ}{=} 1$.