Eilenberg MacLane polynomial functors (2/3)

- $F: \mathcal{C} \longrightarrow k\text{-Mod}$
  $(\mathcal{C}, \odot, 0) + 0$ null.
- $cr_d F: \mathcal{C}^{\times d} \to k\text{-Mod}$
- <u>Def</u>: F poly of $\deg \leq d$ if $cr_{d+1} F = 0$ and $cr_d F \neq 0$

<u>Prop</u>: $F(0) = 0$

$$F(X_1 \odot \cdots \odot X_n) = \bigoplus_{k=1}^{n} \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} cr_k F(X_{i_1}, \dots, X_{i_k}).$$

Let $F(0): \mathcal{C} \to k\text{-Mod}$ be the constant functor

$\forall x \quad F(X) \to F(0)$

$$F(0) \hookrightarrow F \to \overline{F} \to 0$$

$\forall x \quad F(0) \to F(X)$

0 initial.

$$F = F(0) \oplus \overline{F} \quad \text{where } \overline{F}(0) = 0$$

<u>Prop</u>: If $F, G: \mathcal{C} \to k\text{-Mod}$ are poly functors

- $F \oplus G$ is poly and $\deg(F \oplus G) = \max\{\deg F, \deg G\}$
- $F \otimes G$ is poly and $\deg(F \otimes G) \leq \deg F + \deg G$.

<u>Rem</u>: The inequality can be strict

ex: $F_{\mathbb{Q}/\mathbb{Z}} : Ab \to Ab$ $\quad F(G) = G \otimes \mathbb{Q}/\mathbb{Z}$

F is poly of $\deg = 1$ but $F \otimes F = 0$

since $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$

<u>Prop</u>: If k is an integral domain and $F, G : \mathcal{C} \to k\text{-Mod}$ are poly. taking torsion free values

$$\deg(F \otimes G) = \deg F + \deg G.$$

<u>Ex</u>: $Id : k\text{-mod} \to k\text{-Mod}$ is poly of $\deg = 1$

$T^d : k\text{-mod} \to k\text{-Mod}$ is poly of $\deg = d$

<u>Prop</u>: $\mathcal{C} \xrightarrow{F} k\text{-Mod} \xrightarrow{G} k'\text{-Mod}$.

If F, G are poly. then $G \circ F$ is poly and $\deg(G\circ F) = \deg(G)\cdot\deg(F)$

Ex: $gr \xrightarrow{a} ab = \mathbb{Z}\text{-mod} \xrightarrow{T^d} \mathbb{Z}\text{-mod}$

abelianization $\qquad\qquad \overset{\circ}{=} d$

$a^{\otimes d} := T^d \circ a$ is poly of degree d. $\qquad (\overset{\circ}{=} 1)$

Prop: The functor $cr_d : \mathcal{F}(\mathcal{C}, k) \to \mathcal{F}(\mathcal{C}^{\times d}, k)$ is exact $(\forall d \geq 1)$

Def: A full subcategory $\mathcal{C}'$ of an abelian category $\mathcal{C}$ is thick (or is a Serre subcategory of $\mathcal{C}$) if $0 \in \mathcal{C}'$ and $\mathcal{C}'$ is closed under extensions

i.e. for every short exact sequence

$$0 \to B \to A \to C \to 0 \text{ in } \mathcal{C}$$

$$A \in \mathcal{C}' \iff B \text{ and } C \text{ are in } \mathcal{C}'$$

Prop: $\mathcal{P}ol_d(\mathcal{C}, k)$ is a thick subcategory of $\mathcal{F}(\mathcal{C}, k)$ closed by limits and colimits

ex: $\Gamma^d : k\text{-mod} \to k\text{-Mod}$.

$$\Gamma^d(M) = \underbrace{\left(T^d(M)\right)^{\sigma_d}}_{} \hookrightarrow \underbrace{T^d(M)}_{\text{deg } d}$$

$\Rightarrow \Gamma^d$ is poly of deg d.

Ex: The Passi functors (LNM Passi 1979)

$P_d : \text{gr} \to \text{Ab}$ where IG is the augmentation ideal.

$G \mapsto IG/I^{d+1}G$ $(= \ker(\mathbb{Z}G \to \mathbb{Z}))$

$P_1 = a \quad (IG/I^2G \simeq a(G))$

we have exact sequence of abelian grps

$$0 \to I^dG/I^{d+1}G \to IG/I^{d+1}G \to IG/I^dG \to 0$$

giving <u>non-split</u> ses of functors

$$0 \to \underbrace{a^{\otimes d}}_{\text{deg } d} \to P_d \to \underbrace{P_{d-1}}_{\text{deg } d-1} \to 0$$

$d=2$ $\quad 0 \to \underbrace{a^{\otimes 2}}_{\text{deg } 2} \to P_2 \to \underbrace{a}_{\text{deg } 1} \to 0$

P_2 is poly of deg 2

By induction $\underline{\delta_d \text{ is poly of } \leq d.}$

δ_d is $\underline{\text{NOT}}$ of the form $\underbrace{\bigoplus_{i=1}^{d} F(i) \otimes_{\mathfrak{S}_i} I^{\otimes i}}$

for $F : \Sigma \to Ab$

Σ is cat of finite sets and bijection.

$\underline{\text{Rem}}$: In terms of homological algebra.

$$\delta_2 \in \mathrm{Ext}^1_{\underline{\mathcal{F}(gr)}}(I, I^{\otimes 2}) \simeq \mathbb{Z}$$

$\underline{\text{Rem}}$:

$$\begin{tikzcd} Pol_d(gr, \mathbb{Z}) \arrow[r, hook] & \mathcal{F}(gr, \mathbb{Z}) \arrow[l, bend right, "q_d"'] \end{tikzcd}$$

has a left adjoint q_d

$q_d(F)$ is the biggest quotient of F belonging to $Pol_d(gr, Ab)$

$P_1 : gr \to Ab$ $\quad P_1(G) = \mathbb{Z}[gr(\mathbb{Z}, G)] = \mathbb{Z}[G]$

$$P_1 \simeq \underbrace{P_1(0)}_{\simeq \mathbb{Z}} \oplus \overline{P_1}$$

$\overline{P_1}$

$$\cdots \to q_{d+1}(\overline{P_1}) \twoheadrightarrow q_d(\overline{P_1}) \twoheadrightarrow q_{d-1}(\overline{P_1}) \twoheadrightarrow \cdots \quad q_1(\overline{P_1})$$

$$\begin{matrix} \| & \cong & \cong \\ \to S_{d+1} \twoheadrightarrow & S_d \twoheadrightarrow & S_{d-1} \twoheadrightarrow \end{matrix}$$

· Eilenberg MacLane original definition.

$\mathcal{C}$ is an additive category (denoted by $\mathcal{A}$)

$\Rightarrow \forall x, y \in Ob(\mathcal{A}) \quad \mathcal{A}(x, y)$ are abelian gps.

$F : \mathcal{A} \to k\text{-Mod}$

$$\mathcal{A}(x, y) \longrightarrow k\text{-Mod}(F(x), F(y))$$

$$f \longmapsto F(f).$$

F is additive if $\forall x, y \in \mathcal{A}$, $f, g \in \mathcal{A}(x, y)$

$$F(f+g) = F(f) + F(g)$$

$$\Leftrightarrow F(f+g) - F(f) - F(g) = 0 \qquad \text{I}$$

<u>Def</u>: <u>EMcL polynomial function</u>

$(G, +)$, $(V, +)$ abelian gps.

$f : G \to V$ a set map.

d-th deviation of f.

$\mathrm{dev}_d(f) : G^d \to V$

$$\mathrm{dev}_d(f)(x_1, \dots, x_d) := \sum_{1 \le i_1 < \dots < i_r \le d} (-1)^{d-r} f(x_{i_1} + \dots + x_{i_r}).$$

ex: $\mathrm{dev}_1(f)(x_1) = f(x_1)$

$\mathrm{dev}_2(f)(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2)$.

f is poly of $\overset{\circ}{=} d$ if $\mathrm{dev}_{d+1}(f) = 0$
and $\mathrm{dev}_d(f) \neq 0$

Prop: $f(x_1 + \dots + x_d) = \sum \mathrm{dev}_r(f)(x_{i_1}, \dots, x_{i_r})$.

Def: $F : \mathcal{A} \to k\text{-Mod}$ is an EMcL poly function of degree d if

$$\mathcal{A}(x, y) \xrightarrow{F} k\text{-Mod}(F(x), F(y))$$

are poly functions of $\le d$.

ex: $\mathrm{dev}_2(F)(f_1, f_2) = F(f_1 + f_2) - F(f_1) - F(f_2)$

- (strict) polynomial functors

(Friedlander–Suslin 97
Touzé, Krause, ...)

Idea: Replace the notion of EML polynomial functions by polynômes

M, N R-Modules for R commutative ring

Def [Roby 1963]

- A homogeneous polynôme of $\deg d$ from M to N is a R-linear map

$$\Gamma^d(M) \longrightarrow N$$

where
- If M is projective $\Gamma^d(M) = (M^{\otimes d})^{S_d}$
- In general $\Gamma^d(M)$ is generated by symbols $\gamma_d(x)$ + relations.

$$\gamma_d(x) \Updownarrow \text{"}\frac{x^d}{d!}\text{"}$$

$$Pol_d(M,N) := \mathrm{Hom}_{R\text{-Mod}}(\Gamma^d(M), N)$$

$$Pol(M,N) := \bigoplus_{d \in \mathbb{N}} Pol_d(M,N)$$

$$\begin{array}{ccccc} M & \longrightarrow & \Gamma^d(M) & \longrightarrow & N. \\ x & \longmapsto & \gamma_d(x) & & \\ & & \| \left(x^{\otimes d} \text{ if } M \text{ projective} \right) & & \end{array}$$

$$\begin{tikzcd}
\mathrm{Pol}_d(M,N) \arrow[r, hook] \arrow[dr] & \mathrm{Pol}(M,N) \arrow[r] \arrow[dr, "\alpha_{M,N}"'] & N^M \\
& (\mathrm{Pol}_{EML})_{\leq d}(M,N) \arrow[r, hook] & \mathrm{Pol}_{EML}(M,N) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

1) If $\underline{R=\mathbb{Q}}$ and M,N are finite dim

$\alpha_{M,N}$ is a bijection. ||

2) If $R=k$ is a field and $\mathrm{char}(k)=p>0$

$$k \xrightarrow{f} k$$
$$x \longmapsto x^p$$

since $(x+y)^p = x^p + y^p$

$f \in (\mathrm{Pol}_{EML})_1(k,k).$

but f is associated to an element in $\mathrm{Pol}_d(k,k)$

Def: A (strict) poly functor from of degree d from k-mod to k-mod is

- a collect° of k-v.s. $F(V)$
- for $f: U \to V$ linear map $F(f): F(U) \to F(V)$
- linear maps
$$\Gamma^d(\mathrm{Hom}_k(U,V)) \to \mathrm{Hom}(F(U), F(V))$$

+ composition

⚠ A strict poly functor is NOT a functor on k-Mod

$\Gamma^d(k\text{-Mod})$:
$$\Gamma^d(k\text{-Mod})(U,V) := \Gamma^d(\mathrm{Hom}_k(U,V))$$

$$\mathcal{P}(k) = \bigoplus \mathcal{P}_d(k)$$