Polynomials in Action[1]

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We review several examples of how polynomials interact with monoid and category actions. Written at the 2024 Poly at Work Workshop at the Topos Institute.

[1] Inspired by conversations with Matteo Capucci, Harrison Grodin, Sophie Libkind, Toby Smithe, and David Spivak.

1 *Dynamical systems with actions*

To discuss continuous-time dynamics in **Poly**, we need some notion of an $\mathbb{R}$-action. More generally, we would like to consider how dynamical systems in **Poly** may be equipped with the action of a monoid $(M, e, *)$ (in **Set**), another polynomial, or perhaps any category. Here are several ways we may model such a concept with polynomials.

1. We could consider cofunctors $\mathbb{S} \nrightarrow y^M \otimes \mathbb{C}_p$, where y^M is the carrier of the polynomial comonad corresponding to M viewed as a 1-object category, while $\mathbb{C}_p$ is the cofree comonad on a polynomial functor p. We could replace y^M with an arbitrary category. We could also take an appropriate subcategory of $y^M \otimes \mathbb{C}_p$, associating morphisms in $\mathbb{C}_p$ (tuples of directions in p) with appropriate time values in M.
2. We could consider lenses $\varphi\colon Sy^S \to [My, p]$ such that, for every section $\gamma\colon p \to y$, composing φ with $[My, \gamma]$ yields a cofunctor $Sy^S \nrightarrow [My, y] \cong y^M$.[2] We could also consider lenses $Sy^S \otimes My \to p$ that become cofunctors when composed with any section $p \to y$.
3. We could consider p-coalgebras for polynomial endofunctors p on $[y^M, \mathbf{Set}]$. See the next section for more exposition on this.
4. We could consider the category of (y^M, y^M)-bicomodules, the category of $(\mathbb{C}_p, \mathbb{C}_p)$-bicomodules, or perhaps most generally the category of $(\mathbb{C}, \mathbb{C})$-bicomodules; and the morphisms there whose domain is a comonoid. It turns out that a $(\mathbb{C}, \mathbb{C})$-bicomodules comonad carried by a polynomial $\mathbb{D}$ corresponds to a cofunctor $\mathbb{D} \nrightarrow \mathbb{C}$ (in particular, $\mathbb{D}$ itself must be a polynomial comonad).
5. Let S be a smooth manifold and $f\colon \mathbb{R} \to S$ be a smooth map. In the thin double category whose objects are polynomials, vertical arrows are lenses, and horizontal arrows are charts,[3] we can

[2] This idea comes from Smithe's *Open Dynamical Systems as Coalgebras for Polynomial Functors, with Application to Predictive Processing* (2022), Definition 2.1.

[3] Introduced by David Jaz Myers in *Double Categories of Open Dynamical Systems* (2020).

consider squares

$$
\begin{array}{ccc}
\sum_{x\in\mathbb{R}} y^{T_x\mathbb{R}} & \overset{Tf}{\underset{f}{\rightleftarrows}} & \sum_{s\in S} y^{T_s S} \\
\downarrow\uparrow & & \downarrow\uparrow \\
\mathbb{R}y & \Longrightarrow & p
\end{array}
$$

The right lens is our open dynamical system with state space S and interface p. The left lens is vertical and picks out unit tangent vectors on directions. The bottom chart can be considered a trajectory of position and direction pairs in p.[4]

[4] I learned this from Matteo Capucci.

2 *Polynomial functors over sets with monoid actions*

Let $(M, e, *)$ be a monoid (in **Set**). Its associated 1-object category, viewed as a comonoid object in $(\mathbf{Poly}, y, \triangleleft)$, is carried by the polynomial y^M, which we will use to denote this category. Then an ***M-set*** is a functor $X\colon y^M \to \mathbf{Set}$, or a set X with an M-action $\cdot\colon M \times X \to X$ respecting e and $*$. A morphism of M-sets $X \to Y$ is a natural transformation from X to Y as functors $y^M \to \mathbf{Set}$, or an ***M-equivariant map*** $f\colon X \to Y$, satisfying $f(m \cdot x) = m \cdot f(x)$ for all $m \in M$ and $x \in X$. In other words, the category of M-sets and M-equivariant maps can be identified with the functor category $[y^M, \mathbf{Set}]$.

Following Gambino-Kock, we characterize polynomial endofunctors on $[y^M, \mathbf{Set}]$.[5] As a presheaf category, $[y^M, \mathbf{Set}]$ is a topos. In particular, it is complete and locally cartesian closed: given an M-equivariant map $p\colon E \to B$, the functor between slice categories

$$\Delta_p\colon [y^M, \mathbf{Set}]/B \to [y^M, \mathbf{Set}]/E$$

induced by pullback along p has a right adjoint

$$\Pi_p\colon [y^M, \mathbf{Set}]/E \to [y^M, \mathbf{Set}]/B.$$

Composing this on one side with the product functor $\Delta_!\colon [y^M, \mathbf{Set}] \to [y^M, \mathbf{Set}]/E$ sending $X \mapsto (E \times X \xrightarrow{\pi} X)$ and on the other with the forgetful functor $\Sigma_!\colon [y^M, \mathbf{Set}]/B \to [y^M, \mathbf{Set}]$ sending $(Y \to B) \mapsto Y$ yields the polynomial functor corresponding to $p\colon E \to B$:

$$\Sigma_! \Pi_p \Delta_!\colon [y^M, \mathbf{Set}] \to [y^M, \mathbf{Set}],$$

which we will also denote by p. Then the category of polynomial endofunctors on $[y^M, \mathbf{Set}]$, denoted $\mathbf{Poly}_{[y^M, \mathbf{Set}]}$, has these polynomial objects as functors and all natural transformations[6] between them as morphisms. Equivalently (and we will freely switch between the two characterizations), it is the category where

[5] See *Polynomial Functors and Polynomial Monads* (2009) by Gambino and Kock; we follow their notation. All this can be generalized to an arbitrary locally cartesian closed category (with a terminal object and pullbacks).

[6] Gambino-Kock consider only the *cartesian* natural transformations, but we would like to consider all of them.

- an object is an M-equivariant map $p\colon E \to B$;
- a morphism φ from $p\colon E \to B$ to $p'\colon E' \to B'$ consists of
  - an M-equivariant map $\varphi_1\colon B \to B'$;
  - an M-equivariant map $\varphi^\sharp\colon E' \times_{B'} B \to E$, whose domain is the pullback of p' along φ_1.

There is a functor $\mathbf{Poly} \to \mathbf{Poly}_{[y^M, \mathbf{Set}]}$ induced by the functor $\mathbf{Set} \to [y^M, \mathbf{Set}]$ that sends every set to its constant presheaf, i.e. the same set with a trivial action. There is also a functor $\mathbf{Poly}_{[y^M, \mathbf{Set}]} \to \mathbf{Poly}$ that forgets M-actions. Each polynomial in $\mathbf{Poly}_{[y^M, \mathbf{Set}]}$ therefore has an underlying polynomial $\sum_{i \in I} \prod_{a \in A_i} y$, along with an M-action on I and a compatible M-action on $\sum_{i \in I} A_i$.[7]

We can lift many[8] of the usual structures on **Poly** to this setting. In particular, given $p \in \mathbf{Poly}_{[y^M, \mathbf{Set}]}$, we can consider p-coalgebras $S \to p(S)$ for $S \in [y^M, \mathbf{Set}]$, or perhaps[9] equivalently morphisms from $S \times S \to S$ to p in $\mathbf{Poly}_{[y^M, \mathbf{Set}]}$. In **Poly**, these are our open dynamical systems. Taking $M := \mathbb{R}$ may then be a way to discuss open dynamical systems with some notion of time $t \in \mathbb{R}$.

[7] Given $i \in I$ and $a \in A_i$, the M-action must send (i, a) to $(m \cdot i, b)$ for some $b \in A_{m \cdot i}$.

[8] It would be interesting to verify precisely which structures on **Poly** generalize and which do not.

[9] This needs verification (or perhaps it simply isn't immediately obvious to me).