# (Towards a) Fuzzy type theory

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# Outline

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- To develop a language in which to model opinions
- To (begin to) generalize the correspondence between category theory and type theory to a correspondence with enriched category theory on one side
- To obtain another generalization of Martin-Löf type theory

# What is an opinion?

- Logic of propositions
  - Model with complete lattices (posets with all co/limits)
    - Products (coproducts) represent conjunction (disjunction)
    - The terminal object $\top$ (initial object $\bot$) represents the true (false) proposition
  - Write $P \leqslant Q$ to mean that $P$ implies $Q$.
  - $P$ *holds* when $\top \leqslant P$.

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  - Model with up-sets (slices) of lattices.
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  - More generally, we can take a subcategory $E$ of $L$.
- Logic of opinions
  - Model with *fuzzy* lattices and *fuzzy* up-sets
  - Above, we answer "Is $P \leqslant Q$?" or "Does $P$ hold?" with "yes" or "no", i.e., "0" or "1".
  - Now we answer "Is $P \leqslant Q$?" or "Does $P$ hold?" with a value in an ordered monoid, for instance $[0, 1]$.

# What is an opinion?

| Proof irrelevant | Proof relevant |
|---|---|
| Propositions<br>• Posets<br>• Categories enriched in $\{0, 1\}$ | |
| Opinions<br>• Fuzzy posets<br>• Categories enriched in $[0, 1]$ | |

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- Goal: develop the bottom-right box.

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- Previously, opinions were modeled by real-valued vectors.
- Opinion space was some real vector space.
- Modeling things as vectors plugs you in to a lot of computational tools,
- but it's akin to modeling propositional logic as $\{0, 1\}$-valued vector space.
- Want to capture more of the structure with tailor-made algebraic notion.

# Enriched categories

## Booleans

- The natural ordering on the booleans $\mathbb{B} := \{0, 1\}$ forms a category.
- It has a monoidal structure given by multiplication.
- Thus, we can consider a $\mathbb{B}$-enriched category $\mathcal{C}$:
  - a set of objects $\mathrm{ob}(\mathcal{C})$,
  - for each pair $x, y \in \mathrm{ob}(\mathcal{C})$, an object $\mathrm{hom}(x, y)$ of $\mathbb{B}$,
  - for each $x \in \mathrm{ob}(\mathcal{C})$, a point $1 \to \mathrm{hom}(x, y)$
  - for each $x, y, z \in \mathrm{ob}(\mathcal{C})$, a morphism $\circ : \mathrm{hom}(x, y) \cdot \mathrm{hom}(y, z) \to \mathrm{hom}(x, z)$.
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  - such that ...

We can interpret $\hom(x, y)$ as indicating whether or not $x \leqslant y$.

# Enriched categories

## The interval

- The natural ordering on the interval $\mathbb{I} := [0, 1]$ forms a category.
- It has a monoidal structure given by multiplication.
- Thus, we can consider a $\mathbb{I}$-enriched category $\mathcal{C}$:
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  - for each $x \in \mathrm{ob}(\mathcal{C})$, a point $1 = \hom(x, y)$
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  - such that ...

We can interpret $\hom(x, y)$ as indicating **to what extent** $x \leqslant y$.

# Enriched categories

- In general, we can replace $\mathbb{B}$ or $\mathbb{I}$ with any monoidal category, but here we consider only monoidal categories which are posets, i.e., ordered monoids $\mathbb{M}$.
- Then, given an $\mathbb{M}$-enriched category $\mathcal{C}$ (representing a space of opinions) we ask that it has the enriched (fuzzy) versions of all limits and colimits: all weighted limits and colimits.
- Then we consider a network of individuals, each with their own opinion space and opinion that they are communicating, and study dynamics.
    - Encode the network as a graph, and consider a sheaf over it, valued in the category of $\mathbb{M}$-enriched categories.

# Weighted limits and colimits

- In a category, we can consider the product $A \times B$ of two objects, $A$, $B$
- But the concept of 'weighted limits' allows us to weight both $A$ and $B$ by sets $\alpha$ and $\beta$.
- The product with this weighting is then the product of $\alpha$-many copies of $A$ and $\beta$-many copies of $B$ ($A^\alpha \times^\beta B$)
- In a $\mathbb{M}$-enriched category, to take a product of $A$ and $B$, we take weights $\alpha, \beta \in M$.
- Then $A^\alpha \wedge^\beta B$ behaves like a conjunction of $A$ scaled down by $\alpha$ and $B$ scaled down by $\beta$.

# Weighted meets and joins

Let:

- $S =$ “Alice likes strawberry ice cream.”
- $C =$ “Alice likes chocolate ice cream.”
- $B =$ “Alice likes chocolate ice cream better than strawberry ice cream.”
- $\alpha \in [0, 1]$

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Then we can consider:

- ${}^{\alpha}S =$ "Alice likes strawberry ice cream with intensity $\alpha$."
- $B^1 \wedge {}^{\alpha}S =$ "$B$ and ${}^{\alpha}S$".

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We can prove a 'fuzzy modus ponens':

- $(B^1 \wedge {}^{\alpha}S \leqslant C) = \alpha$ and $(B^1 \wedge {}^{\alpha}S \leqslant {}^{\alpha}C) = 1$

# Fuzzy concepts

Let:

- $P =$ "I like the iPhone."
- $Q =$ "I like the Galaxy."
- $R =$ "I like the Pixel."
- $S = \{P, Q, R\}$

# Fuzzy concepts

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- $P =$ "I like the iPhone."
- $Q =$ "I like the Galaxy."
- $R =$ "I like the Pixel."
- $S = \{P, Q, R\}$

- ▷ We can consider the presheaf $\mathbb{M}$-category $[S, \mathbb{M}]$ whose objects are functions $S \to M$.
- ▷ It is the *completion* of $S$ under weighted co/limits.
- ▷ The elements are of the form

$$P^{\alpha} \wedge^{\beta} Q \wedge^{\gamma} R \quad \text{or} \quad ((P, \alpha), (Q, \beta), (R, \gamma))$$

for $\alpha, \beta, \gamma \in [0, 1]$.

# Outline

# Fuzzy type theory (jww Shreya Arya, Greta Coraglia, Sean O'Connor, Hans Riess, Ana Tenório)

- In the last section, we fuzzified propositional logic by seeing it as a part of category theory, and fuzzifying the enrichment from $\mathbb{B}$ to $\mathbb{I}$ or $\mathbb{M}$.
- Now we fuzzify Martin-Löf type theory by a similar route.
- People might have multiple reasons for their opinions, so this seems appropriate.

# Simple type theory

There is an equivalence of categories between simply typed $\lambda$-calculi and cartesian closed categories.

| STLC | CCC |
|---|---|
| type $A$ | object $A$ |
| term $x : A \vdash b(x) : B$ | morphism $b : A \to B$ |
| conjunction $A \wedge B$ | product $A \times B$ |
| implication $A \Rightarrow B$ | exponential $B^A$ |

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To fuzzify this, we consider on the right-hand side $\mathrm{Set}(\mathbb{M})$-enriched categories.

# Fuzzy sets

$\mathrm{Set}(\mathbb{M})$ is the category whose

- objects are pairs $(X, \nu)$ where $X$ is a set and $\nu : X \to M$
- morphisms $(X, \nu) \to (Y, \mu)$ are functions $f : X \to Y$ such that $\nu(x) \leqslant \mu(fx)$ for all $x \in X$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{\nu} \;\leqslant & \downarrow \mu \\ & & M \end{array}$$

It inherits a monoidal structure from the ones on $\mathrm{Set}$ and $\mathbb{M}$:

- $(X, \nu) \otimes (X, \mu) := (X \times Y, \nu \cdot \mu)$
- The monoidal unit is $(*, 1)$.

# Fuzzy categories

**Definition**

A $\mathrm{Set}(\mathbb{M})$-enriched category $\mathcal{C}$ consists of

- a set of objects $\mathrm{ob}(\mathcal{C})$,
- for each pair $x, y \in \mathrm{ob}(\mathcal{C})$, an object $\mathrm{hom}(x, y)$ of $\mathrm{Set}(\mathbb{M})$,
- for each $x \in \mathrm{ob}(\mathcal{C})$, a point $(1, *) \to \mathrm{hom}(x, y)$
  - i.e., an element of $\mathrm{hom}(x, y)$ with value 1
- for each $x, y, z \in \mathrm{ob}(\mathcal{C})$, a morphism $\circ : \mathrm{hom}(x, y) \otimes \mathrm{hom}(y, z) \to \mathrm{hom}(x, z)$.
  - i.e., a function $\circ : \mathrm{hom}(x, y) \times \mathrm{hom}(y, z) \to \mathrm{hom}(x, z)$ such that $|f||g| \leqslant |g \circ f|$
- such that ...

- Now there can be multiple morphisms/reasons of a type/opinion, but each one comes with some intensity.

# Dependent type theory

- We've talked about propositional logic and the simply typed $\lambda$-calculus, and their categorical interpretations.
- Our goal is actually dependent type theory.
  - Proof relevant first-order logic.
  - Types can be indexed by other types, just as predicates in first-order logic are indexed by sets.
  - In propositional logic, we have types/propositions $A$, in simply-types $\lambda$-calculus, we have terms/proofs $x : A \vdash b(x) : B$, and in dependent type theory we have dependent types $x : A \vdash B(x)$.

# Display map categories

**Definition**

A *display map category* is a pair $(\mathcal{C}, D)$ of a category $\mathcal{C}$ and a class $D$ of morphisms (called *display maps*) of $\mathcal{C}$ such that

- $\mathcal{C}$ has a terminal object $*$
- every map $X \to *$ is a display map
- $D$ is stable under pullback

- The objects interpret types, the morphisms interpret terms, and the display maps interpret dependent types, and sections of display maps interpret dependent terms.
- From a dependent type $x : B \vdash E(x)$, we can always form $\vdash \pi : \Sigma_{x:B} E(x) \to B$, and this is represented by the display maps.

# Fuzzy display map categories

**Definition**

A *fuzzy display map category* is a pair $(\mathcal{C}, D)$ of a $\mathrm{Set}(\mathbb{M})$-enriched category $\mathcal{C}$ and a class $D$ of morphisms (called *fuzzy display maps*) of $\mathcal{C}$, each of which has value 1, such that

- $\mathcal{C}$ has a terminal object $*$
- every map $X \to *$ is a display map
- $D$ is stable under particular weighted pullbacks

# Fuzzy terms

- The objects of a fuzzy display map category represent types (or contexts).
- The display maps $d : E \to B$ represent dependent types.
- In non-fuzzy display map categories, terms are represented as sections of display maps. Now our sections are fuzzy.

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**Definition**

An $\alpha$*-fuzzy section* of a fuzzy display map is a section with value at least $\alpha$.

- These represent terms $x : B \vdash s :_\alpha E(x)$.

# Substitution / weighted pullbacks

In the definition of *fuzzy display-map category*, we ask that the class of display maps is stable under particular weighted pullbacks.

$$\begin{array}{ccc} \bullet & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow d \\ A & \xrightarrow{f} & B \end{array}$$

- We choose the weight on $A$ to be the singleton with value 1 and the weight on $B$ to be the singleton with the value of $f$.
- Thus, the vertical maps have the same value (1), as do the horizontal maps.

# Structural rules

$$\frac{}{\vdash \diamond\ \mathtt{ctx}}\ \text{(C-Emp)} \qquad \frac{\Gamma \vdash A\ \mathtt{Type}}{\vdash \Gamma, x{:}A\ \mathtt{ctx}}\ \text{(C-Ext)}$$

$$\frac{\vdash \Gamma, x{:}A, \Delta\ \mathtt{ctx}}{\Gamma, x{:}A, \Delta \vdash x{:}_1 A}\ \text{(Var)} \qquad \frac{\Gamma \vdash s{:}_\alpha A}{\Gamma \vdash s{:}_\beta A}\ \text{(Cons)}$$

$$\frac{\Gamma, \Delta \vdash B\ \mathtt{Type} \quad \Gamma \vdash A\ \mathtt{Type}}{\Gamma, x{:}A, \Delta \vdash B\ \mathtt{Type}}\ (\text{Weak}_{ty}) \qquad \frac{\Gamma, \Delta \vdash b{:}_\beta B \quad \Gamma \vdash A\ \mathtt{Type}}{\Gamma, x{:}A, \Delta \vdash b{:}_\beta B}\ (\text{Weak}_{tm})$$

$$\frac{\Gamma, x{:}A, \Delta \vdash B\ \mathtt{Type} \quad \Gamma \vdash a{:}_\alpha A}{\Gamma, \Delta[a/x] \vdash B[a/x]\ \mathtt{Type}}\ (\text{Subst}_{ty}) \qquad \frac{\Gamma, x{:}A, \Delta \vdash b{:}_\beta B \quad \Gamma \vdash a{:}_\alpha A}{\Gamma, \Delta[a/x] \vdash b[a/x]{:}_\beta B[a/x]}\ (\text{Subst}_{tm})$$

**Theorem**

Fuzzy display map categories validate these rules.

# Future work

## Goals and questions

- Add type formers, like *weighted* conjunction
- Do we want to fuzzify other relations in type theory, like equality?
- Use this to study opinion dynamics

Thank you!