

# Philosophical perspectives on category theory

David Corfield

University of Kent

9 March, 2023

# Outline of talk

- 1 Introduction: philosophy meets category theory
- 2 Initial project
- 3 Applied category theory

1 Introduction: philosophy meets category theory

2 Initial project

3 Applied category theory

# Some philosophical questions

- *What is there?* Metaphysics
- *How do we know?* Epistemology
- *How should we think?* Logic.
- *What should we do?* Ethics/Political Philosophy
- ...

# Some philosophical questions

- *What is there?* Metaphysics
- *How do we know?* Epistemology
- *How should we think?* Logic.
- *What should we do?* Ethics/Political Philosophy
- ...

Plenty of scope to set category theory to the task of answering some of these questions.

Since category theory arose within mathematics, it's natural to wonder what this says about mathematics.

# Approaches to the philosophy of mathematics

People pose philosophical questions to mathematics in different ways:

- I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?

# Approaches to the philosophy of mathematics

People pose philosophical questions to mathematics in different ways:

- I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?
- How should we characterise the **rationality of the process** that has taken us from Euclid to current mathematics? What does progress amount to? How do conceptualizations change? What do we learn from the changing interplay between mathematics and physics?

# Approaches to the philosophy of mathematics

People pose philosophical questions to mathematics in different ways:

- I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?
- How should we characterise the **rationality of the process** that has taken us from Euclid to current mathematics? What does progress amount to? How do conceptualizations change? What do we learn from the changing interplay between mathematics and physics?

I was always far more interested in the latter path, so thought I'd found heaven when back in 1995 I encountered a mathematical physicist writing

...



# John Baez and This Week's Finds

The other, called the “The Tale of  $n$ -Categories”, was an introduction to categories and higher categories — mainly just 2-categories. It can be found in these issues:

- [Week 73](#) - The category of sets and the 2-category of categories.
- [Week 74](#) - Kinds of categories: monoids, groups, and groupoids. The periodic table of  $n$ -categories.
- [Week 75](#) - The fundamental groupoid of a topological space. The classifying space of a groupoid.
- [Week 76](#) - Equations, isomorphisms, and equivalences. Adjoint functors.
- [Week 77](#) - Adjoint functors.
- [Week 78](#) - Adjoint functors and adjoint linear operators.
- [Week 79](#) - The unit and counit of an adjunction.
- [Week 80](#) - The definition of 2-category.
- [Week 83](#) - Adjunctions in 2-categories and dual objects in monoidal categories.
- [Week 84](#) - Review. Monads and monoids.
- [Week 89](#) - Monads in 2-categories. Monoids and monoidal categories as monads.
- [Week 92](#) - Monads from adjunctions.
- [Week 99](#) - 2-Hilbert spaces. Coproducts.
- [Week 100](#) - Definitions of  $n$ -category.

1 Introduction: philosophy meets category theory

2 Initial project

3 Applied category theory

# My project, c. 2000

- What do we learn from the ongoing expansion of (higher) category theory in mathematics and physics?
- What does this shift of foundational language amount to? Have we just been dealing with shadows which we now need to categorify? Why the string diagrams? What is higher gauge theory? Etc.

Our proof of Theorem B is entirely elementary. We analyse the numerical properties of the sequence of numbers  $\zeta_X(n) := |[T^n, X]|$  related to the Euler-Morava characteristics of  $X$  only via Lurie's formula, which we treat as a black-box. Nevertheless, we take the opportunity to provide some theoretical context, that might point towards **deeper structural phenomena of which the results of this paper are mere numerical shadows** (and perhaps also to stronger versions thereof). As this is solely for expository purposes, we merely sketch the relevant ideas in an informal way. More information can be found in [HL13, CSY22, CSY21, Lur19].

Lior Yanovski, [Homotopy Cardinality via Extrapolation of Morava-Euler Characteristics](#)

# My project, c. 2000

- What do we learn from the ongoing expansion of (higher) category theory in mathematics and physics?
- What does this shift of foundational language amount to? Have we just been dealing with shadows which we now need to categorify? Why the string diagrams? What is higher gauge theory? Etc.

Our proof of Theorem B is entirely elementary. We analyse the numerical properties of the sequence of numbers  $\zeta_X(n) := |[T^n, X]|$  related to the Euler-Morava characteristics of  $X$  only via Lurie's formula, which we treat as a black-box. Nevertheless, we take the opportunity to provide some theoretical context, that might point towards **deeper structural phenomena of which the results of this paper are mere numerical shadows** (and perhaps also to stronger versions thereof). As this is solely for expository purposes, we merely sketch the relevant ideas in an informal way. More information can be found in [HL13, CSY22, CSY21, Lur19].

Lior Yanovski, [Homotopy Cardinality via Extrapolation of Morava-Euler Characteristics](#)

- As an experiment, in 2006 I posed the question of how to categorify Klein geometry.
- Instead of quotients of groups by stabilizers (Erlangen program), consider 2-groups and their actions.

# Categorification of geometry

My proposed categorification of [Klein geometry](#):

> Either way what prevents a Erlangen program for 2-groups?

Nothing! Especially since the [Erlangen program](#) is just the flip side of [Galois theory](#): (see especially the slide about the icosahedron), and Galois theory has already been n-categorified to powerful and still growing effect.

But you're right - nobody seems to have thought hard about Klein geometry with Lie 2-groups (or higher) replacing Lie groups. Somehow people have skipped straight to categorifying principal bundles, even though principal bundles are a stripped-down way of thinking about Cartan geometries, which generalize Klein geometries! Sometimes ontogeny fails to recapitulate phylogeny. So, maybe the "punters" should be handed a nice specific Lie 2-group, some 2-spaces on which it acts, and be asked to study the "incidence relations" between these figures. Incidence geometry could be given a whole new lease on life!

Best, jb

# Categorification of geometry

 **September 4, 2006**

## Klein 2-Geometry V

Posted by David Corfield

I had hoped to mark my first appearance in the Café with a striking new contribution to our Klein 2-geometry project. The [project](#) began on my old blog back in May, and you can follow it through its twists and turns over the next 3 monthly instalments. I have enjoyed both participating in a mathematical dialogue and, as a philosopher, thinking about what such participation has to do with a theory of enquiry. The obvious comparison for me is with the fictional dialogue *Proofs and Refutations* written by the philosopher Imre Lakatos in the early 1960s. The clearest difference between these two dialogues is that Lakatos takes the engine of conceptual development to be a process of

conjectured result (perhaps imprecisely worded) - proposed (sketched) proof - suggested counterexample - analysis of proof for hidden assumptions - revised definitions, conjecture, and improved proof,

whereas John, I and other contributors look largely to other considerations to get the concepts 'right'. For instance, it is clear that one cannot get very far without a heavy dose of analogical reasoning, something Lakatos ought to have learned more about from Polya, both in person and through his books.

# Categorification of geometry

Klein 2-geometry and its generalisation to [higher Cartan geometry](#) later bore fruit, in particular by way of Urs Schreiber's work in physics.

It is therefore maybe curious to note that while [Cartan geometry](#) as originating in ([Cartan 23](#)) drew its motivation from the mathematical formulation of the theory of [Einstein gravity](#), higher Cartan geometry is well motivated by higher dimensional [supergravity](#) such as 10d [type II supergravity](#) and [heterotic supergravity](#) as well as [11-dimensional supergravity](#).

# Categorification of logic

 **March 6, 2008**

## Worrying About 2-Logic

Posted by David Corfield

Here's a possible problem for the idea of modal logic as 2-logic.



In ordinary first order logic a model of a theory is a set  $X$ . To an  $n$ -ary predicate of the theory we assign a subset of  $X^n$ , to a constant an element of  $X$ , and so on.

For a given  $X$ , we can derive a Galois correspondence between theories modelled on  $X$  and subgroups of  $X!$ , the permutations of  $X$ , as Todd [shows](#).

Now, in first order modal logic (FoS4) a model of a theory is a sheaf. To show completeness we can stick with bog standard sheaves on topological spaces, as Awodey and Kishida show in their paper [Topology and Modality](#). This combines the topological semantics of propositional modal logic with the set-valued semantics of first-order logic. Necessity relates to taking the interior of subsets of the base space.

 **February 25, 2010**

## (Infinity, 1)-logic

Posted by David Corfield

We're having a chat over [here](#) about what an  $(\infty, 1)$ -logic might look like. The issue is that if we can extract a  $(1)$ -logic from ordinary toposes, shouldn't there be an  $(\infty, 1)$ -logic to be extracted from  $(\infty, 1)$ -toposes. This post originated as an ordinary comment, but as things have gone a little quiet at the Café (10 days without a post!), I thought I'd promote it.



Won't there be a sense in which this internal logic to an  $(\infty, 1)$ -topos will have to be interpretable as a 'logic of space'? If Set is an especially nice topos which being [well-pointed](#) allows us to understand 1-logic internally and externally, we might hope that the well-pointed  $(\infty, 1)$ -topos of  $\infty$ -groupoids,  [\$\infty\$ Gpd](#), does the same for  $(\infty, 1)$ -logic, given  $\infty$ -groupoids are models for spaces.



# Categorification of logic

- The first of these alludes to the (later scrapped) idea of a categorification sequence: propositional, predicate, modal.

# Categorification of logic

- The first of these alludes to the (later scrapped) idea of a categorification sequence: propositional, predicate, modal.
- By the late 2000s, an unexpected (to me) thing that happened was the rise of  $(1; 1)$ -category theory (Joyal, Lurie, etc.).

# Categorification of logic

- The first of these alludes to the (later scrapped) idea of a categorification sequence: propositional, predicate, modal.
- By the late 2000s, an unexpected (to me) thing that happened was the rise of  $(1; 1)$ -category theory (Joyal, Lurie, etc.).
- By 2012, *Homotopy Type Theory* emerged, a structural logic.
- Forget the scary looking ' $(1; 1)$ ' appearing everywhere – things are made simpler by not imposing uniqueness of identity proofs.

# Categorification of logic

- The first of these alludes to the (later scrapped) idea of a categorification sequence: propositional, predicate, modal.
- By the late 2000s, an unexpected (to me) thing that happened was the rise of  $(1; 1)$ -category theory (Joyal, Lurie, etc.).
- By 2012, *Homotopy Type Theory* emerged, a structural logic.
- Forget the scary looking ' $(1; 1)$ ' appearing everywhere – things are made simpler by not imposing uniqueness of identity proofs.
- The boundary of logic and mathematics is altered, e.g.,  $4(S^3) = \mathbb{Z} = 2\mathbb{Z}$  is part of the 'logic'.

# Categorification of logic

- Modalities and then multiple modalities were added, including those for cohesion, cf. my [Modal homotopy type theory](#) (OUP, 2020).

# Categorification of logic

- Modalities and then multiple modalities were added, including those for cohesion, cf. my [Modal homotopy type theory](#) (OUP, 2020).
- It now seems that *modality* is concerned with multiplicity of modes, rather than the envisaged categorification, [Mike Shulman](#):

# Categorification of logic

Modalities and then multiple modalities were added, including those for cohesion, cf. my [Modal homotopy type theory](#) (OUP, 2020).

It now seems that modality is concerned with multiplicity of modes, rather than the envisaged categorification, [Mike Shulman](#):

There's still so much going on here, cf. Mike's recent [Semantics of multimodal adjoint type theory](#).

# High point of the initial project

New logic for a new mathematics for a new physics.



# High point of the initial project

New logic for a new mathematics for a new physics.

Modal HoTT to allow the expression of twisted equivariant differential cohomology on superorbifolds to allow the formulation of M-theory, 'Hypothesis H'.

# High point of the initial project

New logic for a new mathematics for a new physics.

Modal HoTT to allow the expression of twisted equivariant differential cohomology on superorbifolds to allow the formulation of M-theory, '[Hypothesis H](#)'.

For a discussion of why we need to produce the rational trajectory at this moment, see my [Thomas Kuhn, Modern Mathematics and the Dynamics of Reason](#).

# Mathematical prediction

There are very many consequences arising from this program: anomaly cancellation in M-theory, [mysterious triality](#), [Topological Data Analysis](#), etc., and even expectations of where the Cayley distance kernel is positive semidefinite.

# General and particular aspects of Hypothesis H

Why 4, 7, 11, etc? Sometimes we can speak category-theoretically about the particular (e.g., the free such-and-such), but let's turn to the general.

# Quantum modal logic

One recent body of work coming out of this work involves a turn to topological quantum computing and to a [quantum modal logic](#)

# Philosophy of physics potential

At the [Center for Quantum and Topological Systems](#) they're devising a quantum programming language, [QS](#):

Exciting possibility to think about how quantum physics differs from classical physics, how one might conceptualise its emergence.

- 1 Introduction: philosophy meets category theory
- 2 Initial project
- 3 Applied category theory

## Other topics on the blog

Early posts on probability theory; its combination with logic; machine learning; and so on.

John Baez became interested in networks.



## Other topics on the blog

Early posts on probability theory; its combination with logic; machine learning; and so on.

John Baez became interested in networks.

Applied Category Theory today studies causality, probabilistic reasoning, statistics, learning theory, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, systems biology, complex networks, game theory, robotics, quantum computing, ...

## Other topics on the blog

Early posts on probability theory; its combination with logic; machine learning; and so on.

John Baez became interested in networks.

Applied Category Theory today studies causality, probabilistic reasoning, statistics, learning theory, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, systems biology, complex networks, game theory, robotics, quantum computing, ...

Understandable wish to promote new fields of application of category theory, but I doubt that there's any natural separation from the computer science/ physics applications.

# Category theory allowing knowledge transfer

...we should treat the use of categorical concepts as a natural part of transferring and integrating knowledge across disciplines. The restructuring employed in applied category theory cuts through jargon, helping to elucidate common themes across disciplines. (ACT 2018)

# Some philosophical questions

What is there and what is it like? Metaphysics

How do we know? Epistemology

How should we think? Logic.

What should we do? Ethics/Political Philosophy

# Some philosophical questions

What is there and what is it like? Metaphysics

How do we know? Epistemology

How should we think? Logic.

What should we do? Ethics/Political Philosophy

Technical tools that are typically employed in analytic philosophy are first-order logic and modal logic.

Now we have an opportunity to redo what's done but with a different and much richer toolkit, one that continues to evolve in a highly-principled manner. ACT is already contributing.

# Approaches to mathematics, recalled with HoTT

People pose philosophical questions to mathematics in different ways:

I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?

How should we characterise the rationality of the process that has taken us from Euclid to modern mathematics? What does progress amount to? How do conceptualizations change? What do we learn from the changing interplay between mathematics and physics?

# Approaches to mathematics, recalled with HoTT

People pose philosophical questions to mathematics in different ways:

I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?

How should we characterise the rationality of the process that has taken us from Euclid to modern mathematics? What does progress amount to? How do conceptualizations change? What do we learn from the changing interplay between mathematics and physics?

HoTT shows us that mathematics concerns structures. There's no greater or lesser 'ontological commitment' in kind to entities between logic and mathematics, once you admit the propositions and 1.

# Approaches to mathematics, recalled with HoTT

People pose philosophical questions to mathematics in different ways:

I know mathematical truths, such as  $2 + 2 = 4$ . What kind of thing does '2' refer to? How can I know about abstract entities? How can I establish a true mathematical proposition? How can mathematical propositions be applied?

How should we characterise the rationality of the process that has taken us from Euclid to modern mathematics? What does progress amount to? How do conceptualizations change? What do we learn from the changing interplay between mathematics and physics?

HoTT shows us that mathematics concerns structures. There's no greater or lesser 'ontological commitment' in kind to entities between logic and mathematics, once you admit the propositions and 1.

Good to keep a 'dynamic' attitude in mind, however. Who knows what comes next, a directed HoTT? (Note that a lecture series on Univalent Directed Type Theory at [CMU](#) begins next week.)



## Some lines to explore

Quine modi ed: To be is to belong to a typè,  $a : A$ .

## Some lines to explore

Quine modi ed: To be is to belong to a typè,  $a : A$ .

Category theory meets database theory meets metaphysics.

Gilbert Ryle's [category mistakes](#)

"We are in the dark about the nature of philosophical problem and methods if we are in the dark about types and categories."

## Some lines to explore

Quine modi ed: To be is to belong to a typè,  $a : A$ .

Category theory meets database theory meets metaphysics.

Gilbert Ryle's [category mistakes](#)

"We are in the dark about the nature of philosophical problem and methods if we are in the dark about types and categories."

Robert Brandom's inferentialism, types as making explicit preservation of meaningfulness.

# Natural language and type theory/category theory

Aarne Ranta on Type-theoretic grammar

DisCoCat

Categorisation versus placing at a distance in a vector space.

# Natural language and type theory/category theory

Aarne Ranta on Type-theoretic grammar

DisCoCat

Categorisation versus placing at a distance in a vector space.

# Natural language and type theory/category theory

Linear HoTT as a synthesis?

# Lawvere on quantifiers

For  $H$  a topos (or 1-topos) and  $f : X \rightarrow Y$  an arrow in  $H$  induces a 'base change',  $f^*$ , between slices (categories of dependent types):

$$\left( \begin{array}{ccc} X & & Y \\ \downarrow f & \xrightarrow{a} & \downarrow f \\ a & f & a \end{array} \right) : H = X \downarrow f \rightarrow H = Y \downarrow f$$

This base change has dependent sum and product as left and right adjoints

# Common technique

Possible worlds  $W$  !  $V$ , accessibility relation to express possible and necessary types.



# Common technique

Possible worlds  $W$  !  $V$ , accessibility relation to express possible and necessary types.

Temporal type theory:  $b; e$  : Time Interval Time Instant to express operators,  $F; G; H; P$ , then composition for 'until' and 'since'. (Cf. Ch. 4 of [MHoTT](#).)

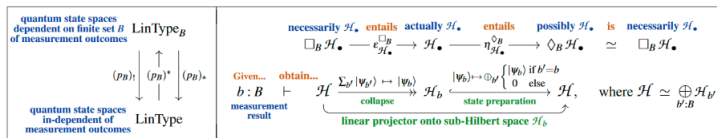
# Common technique

- Possible worlds:  $W \dashv V$ , accessibility relation to express possible and necessary types.
- Temporal type theory:  $b; e : \text{Time Interval} \dashv \text{Time Instant}$  to express operators,  $F; G; H; P$ , then composition for 'until' and 'since'. (Cf. Ch. 4 of [MHoTT](#).)
- Behavioral mereology:  $A \dashv C \dashv B$  to express constraints, allowance and ensurance.
- Equivariance can be built in.
- Integral transforms: push-pull.

# Possible worlds and many worlds

From all this we find, first of all, that

1. the [necessity-count](#) exhibits the [collapse](#) of [quantum states](#) under a [quantum measurement](#) in the  $B$ -basis
2. the [possibility-unit](#) exhibits the condition quantum state preparation in this basis:



**Remark 3.1.** The [collapse](#)-principle of [quantum measurement](#) accurately expressed by the above linear [necessity](#)  $\rightarrow$  actuality  $\rightarrow$  [possibility](#) map was maybe first highlighted and made explicit in [von Neumann \(1932\), §III.3](#), at least for the special case that (in our notation here)  $\mathcal{H}_\bullet = (p_B)^* \mathbb{C}$  (cf. [Lüders \(1951\)](#)).

**Remark 3.2.** We see that the base type  $B$  (4) of [quantum measurement](#) outcomes plays the role of both

1. the [possible worlds](#) of [classical modal logicians](#);
2. the [many worlds](#) of [quantum interpretationists](#).

(...)

# Conclusions

- Category theory continues to reveal its enormous heuristic power.
- Its use in mathematics and physics should change our philosophical understanding of these fields in terms of what they are about and where they're aiming.
- Constructions in new applied areas are of just the kind to interest philosophy (systems, causality, learning,...).
- To any young philosopher listening who wonders whether to learn some category theory: I'm glad I did. Things should be easier for you now, but never underestimate the inertia.