Simplicial delta versus fat delta in higher category theory

Simona Paoli[1]

[1]Department of Mathematics
University of Aberdeen (UK)

Topos Institute Colloquium

Simplicial combinatorics

- Let Δ be the simplicial category. Its objects are finite ordered sets

$$[n] = \{0 < 1 < \cdots < n\}$$

  for integers $n \geq 0$ and its morphisms are non decreasing monotone functions.

- The functor category $[\Delta^{op}, \mathcal{C}]$ is the category of simplicial objects and simplicial maps in $\mathcal{C}$.

Simplicial combinatorics, cont.

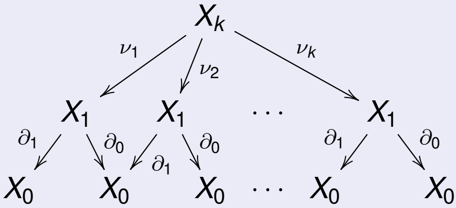
- To give a simplicial object X in $\mathcal{C}$ is the same as to give a sequence of objects $X_0, X_1, X_2, \ldots$ together with face operators $\delta_i : X_n \to X_{n-1}$ and degeneracy operators $\sigma_i : X_n \to X_{n+1}$ $(i = 0, \ldots, n)$ satisfying the simplicial identities.
- We denote $X([n]) = X_n$.

$$X \in [\Delta^{op}, \mathcal{C}] \qquad \cdots X_3 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} X_2 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} X_1 \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} X_0$$

Segal maps.

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

- Let $\mathcal{C}$ be a category with pullbacks. There is a nerve functor

$$N : \mathsf{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

- Given $X \in \mathsf{Cat}\,\mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ for each $k \geq 2$ are isomorphisms.

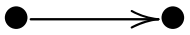
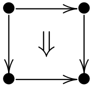
Double categories: some pictures

- Let $X \in \mathsf{Cat}\,(\mathsf{Cat}\,)$

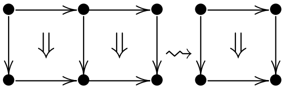
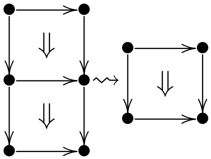
  $X_0 \in \mathsf{Cat}$ has

  objects • morphisms 

  $X_1 \in \mathsf{Cat}$ has

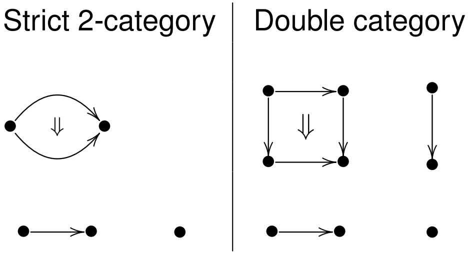
  objects  morphisms 

  Thus squares can be composed horizontally and vertically

  All compositions are associative and unital; interchange law.

Strict 2-categories versus double categories



Thus a strict 2-category is the same as a simplicial object X in Cat such that all the Segal maps are isomorphisms and X_0 is a discrete category.

Definition (P. and Pronk)

$X \in [\Delta^{op}, \mathrm{Cat}]$ is in $\mathrm{Cat}^2_{\mathrm{wg}}$ if

i) The Segal maps are isomorphisms:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

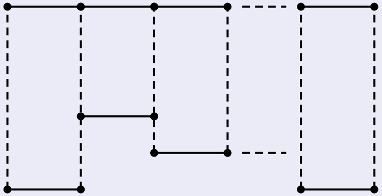
ii) Weak globularity condition: X_0 is an equivalence relation; thus $\gamma : X_0 \to X_0^d$ is an equivalence of categories, where X_0^d is the discrete category on the set of connected components of X_0. We also call X_0 a homotopically discrete category.

iii) The induced Segal maps are equivalences of categories:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \xrightarrow{\simeq} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \qquad k \geq 2$$

Weak globularity condition

- The set underlying X_0^d plays the role of set of objects.
- The induced Segal map condition is equivalent to



Truncation functor and hom category

- Let p : Cat $\to$ Set be the isomorphism classes of objects functor.
- There is a truncation functor

$$p^{(1)} : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Cat}\,,$$

  $(p^{(1)}X)_k = pX_k$ for all $k \geq 0$.
- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, $a, b \in X^d_0$ let $X(a, b)$ be the fibre at (a, b) of

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{(\gamma,\gamma)} X^d_0 \times X^d_0 .$$

Definition

A morphism $F : X \to Y$ in Cat^2_{wg} is a 2-equivalence if

(i) For all $a, b \in X_0^d$ $F(a,b) : X(a,b) \to Y(Fa, Fb)$ is an equivalence of categories.

(ii) $p^{(1)}F$ is an equivalence of categories.

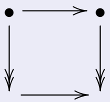
Theorem (P. and Pronk)

$\mathsf{Cat}^2_{\mathsf{wg}}$ *is* 2*-equivalent to bicategories.*

- Given $X \in \mathsf{Cat}^2_{\mathsf{wg}}$ the corresponding bicategory has set of objects X_0^d and hom categories $X(a, b)$ for $a, b \in X_0^d$.

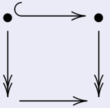
Definition of fat delta

EpiΔ



Objects are epis in Δ and morphisms are commuting squares in Δ.

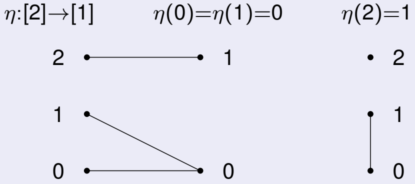
Fat delta $\underline{\Delta}$



Objects are epis in Δ and morphisms are commuting squares in Δ with top map a mono.

A different description of EpiΔ

- Recall that $[n] \in \Delta$ can be seen as a category, the ordinal [n].
- An epi $\eta : [n'] \to [n]$ in Δ identifies a wide subcategory of $[n']$ with morphisms $i \to j$ (for $0 \leq i \leq j \leq n'$) iff $\eta(i) = \eta(j)$.
- The ordinal $[n']$ together with this wide subcategory is called a colored ordinal, and the (non identity) morphisms of the wide subcategory are called colored arrows, and pictured as links.
- *Example:*



A different description of EpiΔ, cont.

- The morphism in EpiΔ

$$\begin{array}{ccc} [n'] & \xrightarrow{\ f\ } & [m'] \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ [n] & \xrightarrow[\ g\ ]{} & [m] \end{array}$$

  preserves colored arrows: if $0 \leq i \leq j \leq n'$ and $\eta_1(i) = \eta_1(j)$, then $\eta_2 f(i) = g\eta_1(i) = g\eta_1(j) = \eta_2 f(j)$.

- Thus EpiΔ is isomorphic to the category whose objects are non-empty colored ordinals and whose morphisms are color-preserving functors between them.

Semi-categories

- A semi-category $\mathcal{C}$ is a diagram in Set

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0$$

  satisfying $d_1 p_2 = d_1 m$, $d_0 p_1 = d_0 m$, $m(\mathrm{Id} \times_{C_0} m) = m(m \times_{C_0} \mathrm{Id})$.

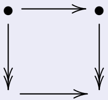
- Let Δ_{mono} be the subcategory of Δ with the same objects and maps the monomorphisms in Δ.

- Δ_{mono} is isomorphic to the category of finite non-empty semi-ordinals (that is, semi-categories associated to a finite total strict order relation).

- We can identify semicategories with functors $X \in [\Delta^{op}_{mono}, \mathrm{Set}]$ in which Segal maps are isomorphisms.

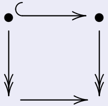
A different description of the fat delta

EpiΔ



Isomorphic to category of finite non empty colored ordinals and color-preserving maps.

Fat delta $\underline{\Delta}$



Isomorphic to category of finite non empty colored semi-ordinals and color-preserving maps.

- Remark: There is a map $\pi : \underline{\Delta} \to \Delta$ given by the target of the surjections.

Vertical and horizontal embeddings

- There are horizontal and vertical inclusions

$$h : \Delta_{mono} \hookrightarrow \underline{\Delta} \qquad v : \Delta_{mono} \hookrightarrow \underline{\Delta}$$

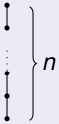
- Given $\varepsilon : [n] \hookrightarrow [m]$ in Δ_{mono}, $h[\varepsilon]$ and $v[\varepsilon]$ are the maps in $\underline{\Delta}$

$$\begin{array}{ccc} [n] & \hookrightarrow & [m] \\ {\scriptstyle \mathrm{Id}_{[n]}} \downarrow & & \downarrow {\scriptstyle \mathrm{Id}_{[m]}} \\ [n] & \hookrightarrow & [m] \end{array} \qquad \begin{array}{ccc} [n] & \hookrightarrow & [m] \\ {\scriptstyle v[n]} \downarrow & & \downarrow {\scriptstyle v[m]} \\ [0] & = & [0] \end{array}$$

- In pictures:

$h[n]$

$v[n]$

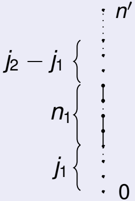


- We will denote $h[n] = [n]$ and $h[\varepsilon] = \varepsilon$.

Segal maps for $X \in [\underline{\Delta}^{op}, \mathcal{C}]$

- Given $\eta : [n'] \twoheadrightarrow [n]$ in $\underline{\Delta}$ let $0 \leq j_1 < j_2 < \cdots < j_t \leq n$ be such that $|\eta^{-1}(j_i)| > 1$ $(i = 1, \ldots, t)$ and let $n_i = |\eta^{-1}(j_i)| - 1$.

- We can picture η as the coloured semi-ordinal

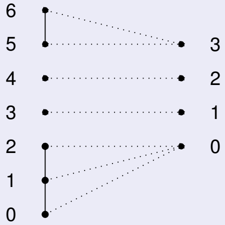


- We have a map in $\mathcal{C}$

$$X_\eta \to X_{j_1} \times_{X_0} X_{v[n_1]} \times_{X_0} X_{j_2 - j_1} \times_{X_0} \cdots X_{v[n_t]} \times_{X_0} X_{n - j_t}$$

Segal map example

- *Example*: for the epi $\eta : [6] \to [3]$



Given $X \in [\underline{\Delta}^{op}, \mathcal{C}]$, we have the Segal map

$$X_\eta \to X_{\nu[1]} \times_{X_0} X_{\nu[1]} \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} X_{\nu[1]} \ .$$

Definition of fair 2-categories

- Recall that a colored category is a category with a specified wide subcategoy whose arrows are called colored arrows.
- $\underline{\Delta}$ is a colored category with coloured arrows the ones sent to identities by $\pi : \underline{\Delta} \to \Delta$.
- Cat is a colored category with coloured arrows the equivalence of categories.

Definition (J. Kock)

A fair 2-category is a color-preserving functor $X : \underline{\Delta}^{op} \to \mathsf{Cat}$ such that X_0 is a discrete category and all the Segal maps are isomorphisms.

Remarks

- Denote

$$\mathcal{O} = X_0 \,, \qquad \mathcal{A} = X_1 \,, \qquad \mathcal{U} = X_{v[1]}$$

  and think of these as categories of objects, arrows, weak identity arrows.

- The Segal maps being isomorphisms gives semicategory structures internal to Cat on $\mathcal{U}$ and $\mathcal{A}$ and a semifunctor

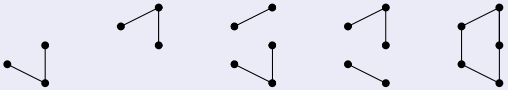
$$\begin{array}{ccccc} \mathcal{U} \times_{\mathcal{O}} \mathcal{U} & \longrightarrow & \mathcal{U} & \rightrightarrows & \mathcal{O} \\ \downarrow & & \downarrow & & \| \\ \mathcal{A} \times_{\mathcal{O}} \mathcal{A} & \longrightarrow & \mathcal{A} & \rightrightarrows & \mathcal{O} \end{array}$$

Remarks, cont.

- The preservation of colors is equivalent to the following maps being equivalences of categories:

$$\mathcal{U} \rightrightarrows \mathcal{O}, \quad \mathcal{U} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} \leftarrow \mathcal{A} \times_{\mathcal{O}} \mathcal{U}, \quad \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}$$

- These correspond to the maps in $\underline{\Delta}$

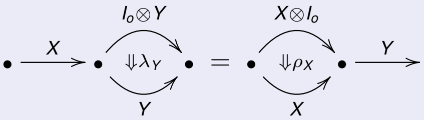


which in fact generate all colored arrows of $\underline{\Delta}$.

- A weak identity arrow (or weak unit) in a semi 2-category $\mathcal{C}$ consists of
  - Arrow $I_o : o \to o$ for $o \in \mathcal{O}b\,\mathcal{C}$
  - Invertible 2-cells (called left and right constraints)

    $$\lambda_Y : I_o \otimes Y \xrightarrow{\sim} Y \qquad \rho_X : X \otimes I_o \simeq X$$

    such that



- A morphism of identity arrows $(I_o, \lambda, \rho) \to (J_o, \lambda', \rho')$ is a 2-cell $I_o \to J_o$ compatible with left and right constraints.

Category of identity arrows

- The category of identity arrows in $\mathcal{C}$ is the disjoint union

$$\mathsf{Id}_{\mathcal{C}} = \coprod_{o \in \mathcal{O}b\,\mathcal{C}} \mathsf{Id}_{\mathcal{C}}(o)$$

Lemma (J. Kock)

The category $\mathrm{Id}_{\mathcal{C}}$ of identity arrows in a bicategory $\mathcal{C}$ is equivalent to the discrete category $\mathcal{O}b\,(\mathcal{C})$.

Proposition (J. Kock)

There is an equivalence of categories

$$\mathsf{Fair}_2 \simeq \mathbb{B}$$

where $\mathbb{B}$ is the category of bicategories with strict composition laws.

- Given a bicategory $\mathcal{C}$ with strict composition laws the corresponding fair 2-category has
  - $\mathcal{O} = \mathcal{O}b\,\mathcal{C}$
  - $\mathcal{A} = \underset{x,y\in\mathcal{O}b\,\mathcal{C}}{\mathrm{Hom}}(x,y) \qquad \mathcal{U} = \coprod_{x\in\mathcal{O}b\,\mathcal{C}} \mathrm{Id}_{\mathcal{C}}(x)$

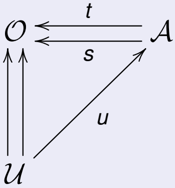
Motivating question

- Both fair 2-categories and weakly globular double categories contain a homotopically discrete category: giving the weak units in the former and the weak globularity condition in the latter.
- Can we interpret the weak globularity condition in terms of weak units?
- To do this, we seek a direct comparison between Fair^2 and $\mathsf{Cat}^2_{\mathsf{wg}}$.
- This involves an interplay between the simplicial delta and the fat delta.

A criterion for fair 2-categories

To give a fair 2-category it is enough to give a discrete category $\mathcal{O}$ and categories $\mathcal{A}$ and $\mathcal{U}$ such that

a) there is a commuting diagram

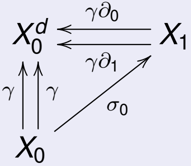


b) $\mathcal{U}$ and $\mathcal{A}$ have semi-category structures internal to Cat (with objects $\mathcal{O}$) such that the following maps are equivalences of categories.

$$\mathcal{U} \rightrightarrows \mathcal{O}, \quad \mathcal{U} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} \leftarrow \mathcal{A} \times_{\mathcal{O}} \mathcal{U}, \quad \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}$$

Using this criterion: preview

- Given $X \in \mathsf{Cat}^2_{\mathsf{wg}}$, set $\mathcal{O} = X_0^d, \quad \mathcal{A} = X_1, \quad \mathcal{U} = X_0$.
- There is a commuting diagram



where $\partial_0, \partial_1 : X_1 \to X_0, \sigma_0 : X_0 \to X_1$, are the structure maps for X.

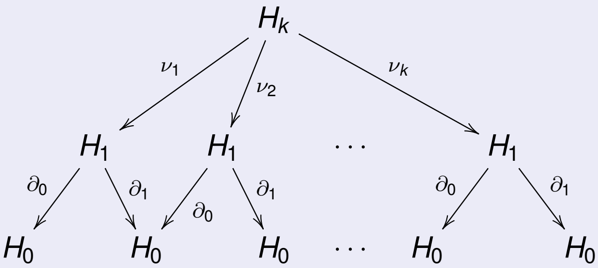
- From the definition of $\mathsf{Cat}^2_{\mathsf{wg}}$, there are equivalences of categories:

$$X_0 \rightrightarrows X_0^d, \quad X_1 \times_{X_0^d} X_0 \to X_1 \leftarrow X_0 \times_{X_0^d} X_1, \quad X_0 \times_{X_0^d} X_0 \to X_0 \ .$$

- It remains to show that X_1 and X_0 have semi-category structures internal to Cat.

Segal maps for pseudo-functors

Let $H \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,]$ be such that H_0 is discrete. The following diagram in Cat commutes, for all $k \geq 2$



Hence there is a unique Segal map for all $k \geq 2$

$$H_k \to H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Definition

The category $\text{SegPs}[\Delta^{op}, \text{Cat}\,]$ is the full subcategory of $\text{Ps}[\Delta^{op}, \text{Cat}\,]$ whose objects H are such that

i) H_0 is discrete.

ii) All Segal maps are isomorphisms: for all $k \geq 2$

$$H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Segalic pseudofunctors and weakly globular double categories

Theorem (P. and Pronk)

a) *There is a functor*

$$Tr_2 : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}\,]$$

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1\,. \end{cases}$$

b) *The strictification functor* $St\ : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,] \to [\Delta^{op}, \mathrm{Cat}\,]$ *restricts to a functor*

$$St\ : \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}\,] \to \mathrm{Cat}^2_{\mathrm{wg}}.$$

- Let $i^* : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Ps}[\Delta^{op}_{mono}, \mathrm{Cat}]$ be induced by $i : \Delta^{op}_{mono} \to \Delta^{op}$.

Proposition

The composite functor

$$\mathrm{Cat}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{i^*} \mathrm{Ps}[\Delta^{op}_{mono}, \mathrm{Cat}]$$

lands in $[\Delta^{op}_{mono}, \mathrm{Cat}]$.

Sketch of proof

- The induced Segal maps ($k \geq 2$)

$$\hat{\mu}_k : X_k = X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 = (Tr_2 X)_k$$

is injective on objects, thus $\nu_k \hat{\mu}_k = \mathrm{Id}$, where ν_k is the pseudo-inverse.

- Thus $Tr_2 \partial_2 : (Tr_2 X)_n \to (Tr_2 X)_{n-1}$ satisfy the semi-simplicial identities. For instance for $k > 2$

$$(Tr_2 X)_{k+1} \xrightarrow{\partial'_j} (Tr_2 X)_k \xrightarrow{\partial'_i} (Tr_2 X)_{k-1}$$

$$\partial'_i \partial'_j = (\hat{\mu}_{k-1} \partial_i \nu_k)(\hat{\mu}_k \partial_j \nu_{k+1}) = \hat{\mu}_{k-1} \partial_i \partial_j \nu_{k+1} =$$

$$= \hat{\mu}_{k-1} \partial_{j-1} \partial_i \nu_{k+1} = (\hat{\mu}_{k-1} \partial_{j-1} \nu_k)(\hat{\mu}_k \partial_i \nu_{k+1}) = \partial'_{j-1} \partial'_i \, .$$

Theorem

There is a functor

$$F_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{Fair}^2$$

such that, for each $X \in \mathsf{Cat}^2_{\mathsf{wg}}$

$$(F_2X)_0 = X_0^d, \qquad (F_1X)_1 = X_1, \qquad (F_2X)_{v[1]} = X_0$$

Proposition

There is a functor

$$T_2 : \mathsf{Fair}^2 \to \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}]$$

such that, for each $X \in \mathsf{Fair}^2$, $(T_2 X)_0 = X_0$, $(T_2 X)_1 = X_1$ *and* $(T_2 X)_r = X_1 \times_{X_0} \overset{r}{\cdots} \times_{X_0} X_1$ *for* $r \geq 2$.

The functor T_2: definition

- For each $\eta : k' \to k$ in $\underline{\Delta}$ and $X \in \mathsf{Fair}^2$ there is an equivalence of categories
$$\alpha_\eta : X_k = X_{\pi(\eta)} \leftrightarrows X_\eta : \beta_\eta$$
such that $\beta_\eta \alpha_\eta = \mathrm{Id}$.

- Given $f : n \to m$ in Δ^{op}, choose $\underline{f} : \eta \to \mu$ in $\underline{\Delta}^{op}$ with $\pi \underline{f} = f$ and let $T_2 f$ be given by the composite
$$X_n \xrightarrow{\alpha_\eta} X_\eta \xrightarrow{\underline{f}} X_\mu \xrightarrow{\beta_\mu} X_m \,.$$

- Is this well-defined?

Lemma

Let $\underline{f} : \eta \to \mu$ *and* $\underline{f}' : \eta' \to \mu'$ *be maps in* $\underline{\Delta}^{op}$ *with* $\pi\underline{f} = \pi\underline{f}'$. *Then, if* $X \in \mathrm{Fair}^2$,

$$\beta_\mu X(\underline{f})\alpha_\eta = \beta_{\mu'} X(\underline{f}')\alpha_{\eta'}.$$

- It follows that $T_2 f$ is well-defined.

The functor T_2: definition, cont.

- Given $n_1 \xrightarrow{f_1} n_2 \xrightarrow{f_2} n_3$ in Δ^{op}, to define $T_2(f_1 f_2)$ we take composable maps in $\underline{\Delta}^{op}$

$$\mu_1 \xrightarrow{\underline{f}_1} \mu_2 \xrightarrow{\underline{f}_2} \mu_3,$$

  such that $\pi(\underline{f}_1) = f_1, \pi(\underline{f}_2) = f_2$.

- We then define $T_2(f_1 f_2)$ to be the composite

$$X_n \xrightarrow{\alpha_{\mu_1}} X_{\mu_1} \xrightarrow{\underline{f}_1 \underline{f}_2} X_{\mu_2} \xrightarrow{\beta_{\mu_2}} X_m.$$

- Why do composable liftings $\underline{f}_1, \underline{f}_2$ of f_1 and f_2 exist ?

Lifting sequences of maps in delta

Proposition

Given maps in Δ

$$n_1 \xrightarrow{f_1} n_2 \xrightarrow{f_2} n_3 \rightarrow \cdots \xrightarrow{f_k} n_{k+1}$$

there are maps in $\underline{\Delta}$

$$\mu_1 \xrightarrow{\underline{f}_1} \mu_2 \xrightarrow{\underline{f}_2} \mu_3 \rightarrow \cdots \xrightarrow{\underline{f}_k} \mu_{k+1}$$

with $\pi \underline{f}_j = f_j$*.*

Comparison result

Definition

Let $R_2 : \mathrm{Fair}^2 \to \mathrm{Cat}^2_{\mathrm{wg}}$ be the composite

$$\mathrm{Fair}^2 \xrightarrow{T_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}\ ,$$

Theorem (P.)

The functors

$$F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \rightleftarrows \mathrm{Fair}^2 : R_2$$

induce an equivalence of categories after localization with respect to the 2-equivalences

$$\mathrm{Cat}^2_{\mathrm{wg}}/\sim\ \simeq\ \mathrm{Fair}^2/\sim\ .$$

Higher dimensions

- Simplicial models of higher categories do exist: for instance weakly globular *n*-fold categories $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ satisfy the homotopy hypothesis and are equivalent to Tamsamani *n*-categories (P.)

- Note that $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}} \hookrightarrow [\Delta^{n-1^{op}}, \mathsf{Cat}\,]$, $\quad \Delta^{n-1^{op}} = \Delta^{op} \times \overset{n-1}{\cdots} \times \Delta^{op}$.

- Question: could we do higher category theory using $\underline{\Delta}^{n-1^{op}} = \underline{\Delta}^{op} \times \overset{n-1}{\cdots} \times \underline{\Delta}^{op}$?

Fair n-categories

- Joachim Kock defined Fair *n*-categories

$$\mathrm{Fair}^{\mathrm{n}} \hookrightarrow [\underline{\Delta}^{n-1^{op}}, \mathrm{Cat}]$$

  encoding strict composition but weak units.

- It is not known if $\mathrm{Fair}^{\mathrm{n}}$ satisfies the homotopy hypothesis.
- Conjecture: $\mathrm{Fair}^{\mathrm{n}}$ is equivalent to $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$.
- This would imply that $\mathrm{Fair}^{\mathrm{n}}$ satisfies the homotopy hypothesis, and is thus a model of weak *n*-categories (Simpson's conjecture).

Summary

- Several models of weak 2-categories.
- Direct comparison between weakly globular double categories and fair 2-categories.
- New light on weakly globular double categories, as encoding weak units.
- New properties of $\underline{\Delta}$, such as lifting of strings of composable maps from Δ to $\underline{\Delta}$.
- Potential for higher dimensional generalisations (proof of weak units conjecture).