Higher Topos Theory and Goodwillie Calculus (Part 1)

André Joyal

UQÀM

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Joint work with

- Mathieu Anel
- Georg Biedermann
- Eric Finster

A very brief history

Modern mathematics essentially began with Cantor's set theory.
The conceptual fabric of modern mathematics is presently moving:

1. Set theory
2. Formal logic
3. Category theory
4. Topos theory
5. Categorical logic
6. Abstract Homotopy theory
7. Higher category theory
8. Higher topos theory
9. Homotopy type theory
10. Brave new mathematics?

Brave new mathematics

David Hilbert:

"From the paradise, that Cantor created for us, no-one shall be able to expel us!"

Norman Steenrod:

"It is the most trivial paper I ever read, and it has the greatest influence on my work!".

John Greenlees:

"The phrase 'brave new rings' was coined by Friedhelm Waldhausen, presumably to capture both an optimism about the possibilities of generalizing rings to ring spectra, and a proper awareness of the risk that the new step in abstraction would take the subject dangerously far from its justification in examples."

Simplicial sets

The category Δ

$Ob(\Delta) := \{[n] = \{0, \ldots, n\} | n \geq 0\}$

$Hom([m], [n])$ is the set of order preserving maps $[m] \to [n]$.

A *simplicial set* is a presheaf $A : \Delta^{op} \to \mathrm{Set}$.
Notation: $A_n = A([n])$.

$$\mathrm{sSet} := Fun(\Delta^{op}, \mathrm{Set})$$

Example: $\Delta[n] = Hom(-, [n])$

The geometric realisation

The *realisation functor* $R : \Delta \to \mathrm{Top}$ is defined by letting

$$R[n] := \{(x_1, \dots, x_n) \in [0,1]^n_{\mathbb{R}} \mid x_1 \leq \dots \leq x_n\}$$

The "*singular complex*" of a topological space X is then defined by

$$S(X)_n := \mathrm{Top}(R[n], X)$$

The functor $S := R^\star : \mathrm{Top} \to \mathrm{sSet}$ has a left adjoint $R_! : \mathrm{sSet} \to \mathrm{Top}$ called the *geometric realisation functor*.

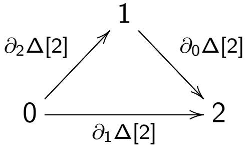
By construction,

$$R_!(A) = \int^{[n]\in\Delta} A_n \times R([n])$$

On Kan complexes

Recall that the *fundamental simplex* $\Delta[n] \in \text{Set}$ is the presheaf $Hom(-,[n]) : \Delta^{op} \to Set$.

The simplex $\Delta[n]$ has *faces* $\partial_i \Delta[n] \subset \Delta[n]$ $(0 \leq i \leq n)$.



and a *boundary*

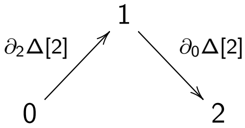
$$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial_i \Delta[n]$$

On Kan complexes

Recall that the *horn* $\Lambda^k[n] \subset \Delta[n]$ $(0 \leq k \leq n)$ is defined by putting

$$\Lambda^k[n] = \bigcup_{i \neq k} \partial_i \Delta[n]$$

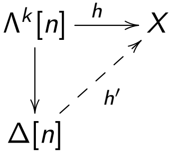
For example, $\Lambda^1[2]$ is



Kan complexes

Definition

A simplicial set $X \in \Delta Set$ is called a *Kan complex* if every horn $h : \Lambda^k[n] \to X$ has a filler $h' : \Delta[n] \to X$.



Theorem

[Quillen] *The category of simplicial set* sSet *admits a cartesian closed Quillen model structure in which the cofibrations are the monomorphisms and the fibrant objects are the Kan complexes.*

We shall say that a Kan complex is a *space*.

The fundamental category

A variation on geometric realisation.

The *nerve* $N(C)$ of a category C is defined by letting

$$N(C)_n := i^\star(C)_n = Fun(i[n], C)$$

where $i : \Delta \subset \mathrm{Cat}$ is the inclusion functor.

The functor $N := i^\star : \mathrm{Cat} \to \mathrm{sSet}$ has a left adjoint $\tau_1 : \mathrm{sSet} \to \mathrm{Cat}$ called the *fundamental category functor*. By construction,

$$\tau_1(A) = \int^{[n]\in\Delta} A_n \times i([n])$$

The *fundamental groupoid* of A is the groupoid reflection of $\tau_1(A)$.

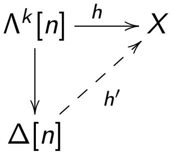
On quasi-categories [BV]

We say that a horn $\Lambda^k[n] \subset \Delta[n]$ is *inner* if $0 < k < n$.

The following notion was introduced by Boardman and Vogt without a name (it is often called a *weak Kan complex*).

Definition

[BV] A simplicial set X is called a *quasi-category* if every inner horn $h : \Lambda^k[n] \to X$ has a filler $h' : \Delta[n] \to X$.



Every Kan complex is a quasi-category.

The nerve $N(C)$ of a small category C is a quasi-category.

On quasi-categories

Boardman and Vogt introduces the *homotopy category* $ho(X)$ of a quasi-category X. It happens that $ho(X) = \tau_1 X$.

Lemma

[J] *A quasi-category X is a Kan complex if and only if its homotopy category $ho(X)$ is a groupoid.*

On quasi-categories

Theorem

[J] *The category of simplicial set* sSet *admits a cartesian closed Quillen model structure in which the cofibrations are the monomorphisms and the fibrant objects are the quasi-categories.*

If X is a quasi-category, then so is the simplicial set X^A for any simplicial set A.

If X is a quasi-category, then a vertex $a \in X_0$ is said to be an *object* of X and an arrow $f \in X_1$ is said to be a *morphism* $f : d^1(f) \to d^0(f)$.

The hom spaces of a quasi-category

If X is a quasi-category, then so is the simplicial set $X^{[1]} := X^{\Delta[1]}$.

The *hom space* $X(a,b)$ between two objects $a, b \in X_0$ is defined by the following pullback square (of simplicial sets)

$$\begin{array}{ccc} X(a,b) & \longrightarrow & X^{[1]} \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ 1 & \xrightarrow{(a,b)} & X \times X \end{array}$$

The simplicial set $X(a,b)$ is a Kan complex (it is a "space")

Composition in a quasi-category

If X is a quasi-category, then so is the simplicial set $X^{[2]} := X^{\Delta[2]}$.

The generalised *hom space* $X(a,b,c)$ for three objects $a, b, c \in X_0$ is defined by the following pullback square (of simplicial sets)

$$\begin{array}{ccc} X(a,b,c) & \longrightarrow & X^{[2]} \\ \downarrow & & \downarrow {\scriptstyle (d^2,d^1,d^0)} \\ 1 & \xrightarrow{(a,b,c)} & X \times X \times X \end{array}$$

The projection $(d^2, d^0) : X(a,b,c) \to X(a,b) \times X(b,c)$ has a section $s : X(a,b) \times X(b,c) \to X(a,b,c)$.

The composition operation

$$\mu := d^2 s : X(a,b) \times X(b,c) \to X(a,c)$$

is well defined up to homotopy.

Truncated quasi-categories

A quasi-category X is said to be 1-*truncated* if the hom space $X(a, b)$ is 0-truncated for every $a, b \in X_0$.

A quasi-category X is equivalent to a category if and only if it is 1-truncated if and only if the canonical map $X \to ho(X)$ is an equivalence of quasi-categories.

Ordinary category theory is the theory of *1-truncated* quasi-categories.

Brave new category theory

The initial theory [J]:

- Functors and natural transformations;
- The opposite quasi-category;
- Left and right fibrations;
- The slice X/a and the coslice $a\backslash X$ of a quasi-category X by an object $a \in X$.
- Initial and terminal objects;
- Diagrams, limits and colimits;
- Localizations;
- Yoneda lemma (first version);
- Adjoint functors (first version).

The quasi-category of spaces $\mathcal{S}$

A quasi-category X is a Kan complex, if every arrow in X is invertible, in which case we shall say that X is a *homotopy type*, or a *space*.

The quasi-category of spaces $\mathcal{S}$ was constructed by Lurie in [HTT]. The quasi-category $\mathcal{S}$ is large but locally small. It is cocomplete and freely generated by its terminal object $1 \in \mathcal{S}$.

The coslice $1\backslash\mathcal{S}$ is the *quasi-category of pointed spaces* $\mathcal{S}_\bullet$.

It was proved later by Cisinski [C] that the projection $p : \mathcal{S}_\bullet \to \mathcal{S}$ is a universal left fibration: for any left fibration $f : X \to A$ there exists a (homotopy) pullback square

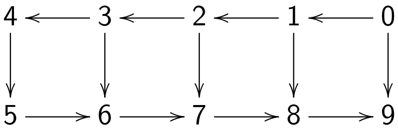
$$\begin{array}{ccc} X & \xrightarrow{c_\bullet} & \mathcal{S}_\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{c} & \mathcal{S} \end{array}$$

and the pair of maps $(c, c_\bullet)$ is homotopy unique.

The twisted category of arrows

The *twisted category of arrows* $T(C)$ of a category C is the category of elements of the functor $Hom : C^{op} \times C \to Set$.

A chain $[n] \to T(C)$ is a functor $[n]^{op} \star [n] \to C$.



$T(X)$ can be defined for any simplicial set X.

By definition $T(X)_n = X([n]^{op} \star [n]) = X_{2n+1}$.

The simplicial set $T(X)$ is a quasi-category when X is a quasi-category.

The Yoneda map

If X is a quasi-category, then the canonical map

$$(s,t) : T(X) \to X^{op} \times X$$

is a left fibration.

It has a classifying map $hom : X^{op} \times X \to \mathcal{S}$

$$\begin{array}{ccc} T(X) & \xrightarrow{hom_\bullet} & \mathcal{S}_\bullet \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle p} \\ X^{op} \times X & \xrightarrow{hom} & \mathcal{S} \end{array}$$

From the map $hom : X^{op} \times X \to \mathcal{S}$ we obtain *the Yoneda map*

$$y : X \to \mathcal{S}^{X^{op}}$$

Remark on pushouts and pullbacks

In category theory, the notions of pushout and of pullback squares depend on the ambiant category. This is also true in the theory of quasi-categories.

Pushouts and pullbacks in $\mathcal{S}$ are *homotopy pushouts and pullbacks*.

For example, the square on the left is a pushout in the category of sets Set

$$\begin{array}{ccc} 1 \sqcup 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array} \qquad\qquad \begin{array}{ccc} 1 \sqcup 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

while the square on the right is a pushout in the quasi-category of spaces $\mathcal{S}$ (where S^1 is the homotopy type of the circle). The square on the left is obtained by applying the functor π_0 to the square in the right.

Lurie's contributions [HTT] and [HA]

Lurie's terminology: quasi-category $\to$ ∞-category.

- Cartesian fibrations;
- The ∞-category of spaces $\mathcal{S}$;
- Yoneda lemma (second version);
- The $(\infty, 2)$-category of small ∞-categories;
- Left and right Kan extensions;
- Presentable ∞-categories;
- ∞-topoi;
- Stable ∞-categories;
- ∞-operads;
- Monads, monadic functors;
- Monoidal ∞-categories, $\mathbb{E}_n$-categories.

On large and small ∞-categories

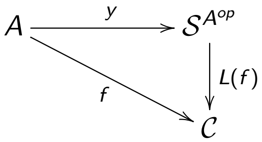
The ∞-category of spaces $\mathcal{S}$ is large and locally small.

The ∞-category of small ∞-categories Cat_∞ is large and locally small.

The ∞-category of large ∞-categories CAT_∞ is very large and not locally small.

The ∞-category of finite spaces Fin is small.

(Lurie) If A is a small ∞-category, then the ∞-category $\mathcal{S}^{A^{op}}$ is cocomplete and freely generated by the Yoneda map $y : A \to \mathcal{S}^{A^{op}}$. More precisely, for every cocomplete ∞-category $\mathcal{C}$ and every functor $f : A \to \mathcal{C}$ there exists a unique cocontinuous functor $L(f) : \mathcal{S}^{A^{op}} \to \mathcal{C}$ such that the following triangle commutes:



Rezk descent principle

The ∞-category $\mathcal{S}$ has a very surprising property which was discovered by Charles Rezk [Rez2].

Consider the contravariant functor $Slice : \mathcal{S}^{op} \to \mathbf{CAT}_\infty$ which takes an object $A \in \mathcal{S}$ to the ∞-category $\mathcal{S}/A$ and which takes a map $f : A \to B$ to the base change functor $f^\star : \mathcal{S}/B \to \mathcal{S}/A$.

Rezk descent principle: The slice functor

$$Slice : \mathcal{S}^{op} \to \mathbf{CAT}_\infty$$

takes colimits to limits:

$$\mathcal{S}/\varinjlim_{i \in I} A_i = \varprojlim_{i \in I} \mathcal{S}/A_i$$

for every diagram $A : I \to \mathcal{S}$.

The descent principle

By the descent principle, we have

$$\mathcal{S}/(A \sqcup B) = \mathcal{S}/A \times \mathcal{S}/B$$

and more generally,

$$\mathcal{S}/\bigsqcup_{i \in I} A_i = \prod_{i \in I} \mathcal{S}/A_i$$

Every space $A \in \mathcal{S}$ is a coproduct of singletons: $A = A \times 1 = \sqcup_A 1$.
By the descent principle, we have

$$\mathcal{S}/A = \mathcal{S}/\sqcup_A 1 = \prod_A \mathcal{S}/1 = \prod_A \mathcal{S} = \mathcal{S}^A$$

∞-topoi

Recall that the category of sets Set is the basic example of a Grothendieck topos. Another example is the category of presheaves $\mathrm{Set}^{C^{op}}$ on a small category C. Every Grothendieck topos $\mathcal{E}$ is a left exact localization of a presheaf category.

Note: a functor is said to be *left exact* if it preserves finite limits.

The quasi-category of spaces $\mathcal{S}$ is the basic example of an ∞-topos. Another example is the quasi-category of presheaves $\mathcal{S}^{C^{op}}$ on a small quasi-category C. Every ∞-topos $\mathcal{E}$ is a left exact localization of a presheaf quasi-category [HTT].

Definition

An *algebraic morphism* of ∞-topoi is a left exact cocontinuous functor $\phi : \mathcal{E} \to \mathcal{F}$.

The descent principle

Lurie's theorem [HTT]:

Theorem

A presentable ∞-category $\mathcal{E}$ is an ∞-topos if and only if the descent principle holds in $\mathcal{E}$:

$$\mathcal{E}/\varinjlim_{i \in I} A(i) = \varprojlim_{i \in I} \mathcal{E}/A(i)$$

for any diagram $A : I \to \mathcal{E}$.

Sheaves

Let $\mathcal{E}$ be an ∞-topos.

We denote by $\Delta(u) : A \to A \times_B A$ the diagonal of map $u : A \to B$ in $\mathcal{E}$ and by $\Delta^n(u)$ the n-th iterated diagonal of u.

The *diagonal closure* of a set of maps $\Sigma \subseteq \mathcal{E}$ is defined to be the set $\Delta^\infty(\Sigma) = \{\Delta^n(u) \mid u \in \Sigma, n \in \mathbb{N}\}$.

Recall that an object $X \in \mathcal{E}$ is said to be *local* with respect to a map $u : A \to B$ if the map $Map(u, X) : Map(B, X) \to Map(A, X)$ is invertible.

Definition

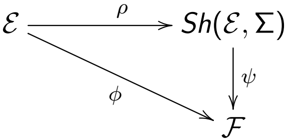
[ABFJ3] Let Σ be a set of maps in an ∞-topos $\mathcal{E}$. We say that an object $X \in \mathcal{E}$ is a Σ-*sheaf* if it is local with respect to every base change of the maps in $\Delta^\infty(\Sigma)$. We write $Sh(\mathcal{E}, \Sigma)$ for the full-subcategory of Σ-sheaves.

Sheaves

Theorem

[ABFJ3] *Let* Σ *be a set of maps in an* ∞*-topos* $\mathcal{E}$*. Then the subcategory* $Sh(\mathcal{E}, \Sigma)$ *of* Σ*-sheaves is reflective and the reflector* $\rho : \mathcal{E} \to Sh(\mathcal{E}, \Sigma)$ *is left exact. The subcategory* $Sh(\mathcal{E}, \Sigma)$ *is an* ∞*-topos and the reflector* ρ *inverts the maps in* Σ *universally among algebraic morphisms of* ∞*-topoi.*

In other words, if $\phi : \mathcal{E} \to \mathcal{F}$ is an algebraic morphism of ∞-topoi and the maps in $\phi(\Sigma) \subseteq \mathcal{F}$ are invertible, then there exists a unique algebraic morphism of ∞-topoi $\psi : Sh(\mathcal{E}, \Sigma) \to \mathcal{F}$ such that $\psi\rho = \phi$.



Topo-logy [AJ]

Topos $=$Logos^{op}
Logos $=$Topos^{op}

By definition, an object of the category $\mathbf{Log}_\infty$ is an ∞-topos (now called an ∞-logos) and a morphism of ∞-logoi $\phi : \mathcal{E} \to \mathcal{F}$ is a left exact cocontinuous functor.

The category of ∞-*topoi* $\mathbf{Top}_\infty$ is defined to be the opposite of the category $\mathbf{Log}_\infty$.

Remark: The category $\mathbf{Log}_\infty$ is actually an $(\infty, 2)$-category, in which the 2-cells are natural transformations.

The category $\mathbf{Log}_\infty$ has many properties in common with the category of commutative rings [HTT] [AJ].

Logos theory vs commutative algebra [HTT][ABFJ5]

commutative ring	**logos**
ring of integers $\mathbb{Z}$	the logos of spaces $\mathcal{S}$
sum: $a + b$	colimit: $A \sqcup_C B$
product: $a \times b$	finite limit: $A \times_C B$
distributive law: $a \times (b + c) = a \times b + a \times c$	base change $u^\star : \mathcal{E}/B \to \mathcal{E}/A$ preserves colimits

Commutative algebra	**Theory of logoi**
morphism of rings $\phi : A \to B$	morphism of logoi $\phi : \mathcal{E} \to \mathcal{F}$
polynomial ring $\mathbb{Z}[x]$	free logos $\mathcal{S}[X]$
tensor product $A \otimes B$	tensor product $\mathcal{E} \otimes \mathcal{F}$
ideal $J \subseteq R$	congruence $\mathcal{J} \subseteq \mathcal{E}$
quotient ring $\rho : R \to R/J$	left exact localization $\rho : \mathcal{E} \to \mathcal{E} /\!\!/ \mathcal{J}$
product of ideals $J_1 \cdot J_2 \subseteq R$	product of congruences $\mathcal{J}_1 \cdot \mathcal{J}_2 \subseteq \mathcal{E}$

Congruences

If $\mathcal{M}$ is a class of maps in an ∞-category $\mathcal{E}$ let us denote by $\tilde{\mathcal{M}}$ the full subcategory of $\mathcal{E}^{[1]}$ spanned by the maps in $\mathcal{M}$.

Definition

[ABFJ3] If $\mathcal{E}$ is an ∞-logos, we say that a class of maps $\mathcal{J} \subseteq \mathcal{E}$ is a *congruence* if the following conditions hold:

1. every isomorphism belongs to $\mathcal{J}$;
2. the class $\mathcal{J}$ is closed under composition;
3. the full-subcategory $\tilde{\mathcal{J}} \subseteq \mathcal{E}^{[1]}$ is a sub-logos ($=$ it is closed under colimits and finite limites)

For example, if $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of ∞-logoi, then the class $\mathcal{J} = \phi^{-1}(\text{Iso})$ is a congruence.

Product of congruences

The category of arrows $\mathcal{E}^{[1]}$ of an ∞-logos $\mathcal{E}$ has a natural symmetric monoidal structure given by the pushout products of maps in $\mathcal{E}$.

Recall that the *pushout product* $f \square g$ of two maps $f : A \to B$ and $g : C \to D$ is the map

$$(A \times D) \sqcup_{A \times C} (B \times C) \to B \times C$$

The *product* of two congruences $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{J}$ is defined in $[\mathrm{ABFJ5}]$ by letting

$$\mathcal{J}_1 \cdot \mathcal{J}_2 = \{f_1 \square f_2 \mid f_1 \in \mathcal{J}_1, f_2 \in \mathcal{J}_2\}^{\mathbf{c}}$$

where $(-)^{\mathbf{c}}$ denotes the congruence closure of a class of maps.

Bibliography

- [ABFJ6] M. Anel, G. Biedermann, E. Finster, A. Joyal. *Unifying Goodwillie and Weiss calculi*. (in preparation)
- [ABFJ5] *Left exact localizations of higher topoi III: the acyclic product*.(submitted)
- [ABFJ4] *Left exact localizations of higher topoi II: Grothendieck topologies*. JPAA (June 2023).
- [ABFJ3] *Left exact localizations of higher topoi I: higher sheaves*. Adv. Math. (2022).
- [ABFJ2] *A generalised Blakers-Massey theorem*. Journal of Topology 13 (2020).
- [ABFJ1] *Goodwillie's calculus of functors and higher topos theory*. Journal of Topology 11.4 (2018).
- [AJ] M. Anel & A. Joyal, *Topo-logie*. A chapter in "New spaces in Mathematics". Camb. UP (2021).

Bibliography

- [AL18] M. Anel and D. Lejay. *Exponentiable higher toposes.* arXiv : 1802.10425v1
- [AS20] M. Anel & L. Subramaniam. *Small object arguments, plus construction, and left exact localizations* arXiv : 2004.00731v2
- [BD] G. Biedermann and B. Dwyer. *Homotopy nilpotent groups.* Algebraic and Geometric Topology (2008)
- [BV] M. Boardman & R. Vogt. *Homotopy invariant algebraic structures on topological spaces.* SLNM 347 (1973)
- [C] D.C. Cisinski. *Higher categories and homotopical algebra.* Cambridge UP (2019).
- [J] A. Joyal. *Quasi-categories and Kan complexes.* JPAA. Vol 175 (2002).
- [HTT] J. Lurie. *Higher topos theory.* (version 10, 2012).
- [HA] J. Lurie. *Higher algebra.*
- [Rez1] C. Rezk. *A model for the homotopy theory of homotopy theories* Trans. AMS Vol.353 (2001)
- [Rez2] C. Rezk. *Toposes and Homotopy toposes* (notes).