Cauchy Completeness and Adjoints in Double Categories

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Motivation

1. Lawvere (1973): Cauchy completeness by considering a (generalized) metric space as a category enriched in $[0, \infty]$.

2. Paré (2021): Cauchy completeness in double categories, and showed an (S, R)-modules M has a right adjoint in $\mathbb{R}\mathrm{ing}$ of commutative rings iff it is finitely generated and projective over S.

3. N./Wood (2017): $- \otimes_S M$ on S-Mod has a left adjoint iff M is fg projective over S, for commutative rings, rigs, (and quantales).

Goals:

- More examples in double categories $(\mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}, \mathbb{T}\mathrm{op}, \mathbb{Q}\mathrm{uant})$.
- Remove commutativity from 3. and relate it directly to 2.

Double Categories

A double category $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\bullet} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (\mathbb{D}_0 \xrightarrow{\mathrm{id}^\bullet} \mathbb{D}_1)$$

Objects X of $\mathbb{D}_0$, called objects of $\mathbb{D}$

Morphisms $X \xrightarrow{f} Y$ of $\mathbb{D}_0$, called horizontal morphisms of $\mathbb{D}$

Objects $X_s \overset{v}{\bullet\!\!\rightarrow} X_t$ of $\mathbb{D}_1$, called vertical morphism of $\mathbb{D}$

Morphisms $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow^{\bullet} & \varphi & \downarrow^{\bullet}{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$ of $\mathbb{D}_1$, called cells of $\mathbb{D}$

A cell is special if f_s and f_t are identity morphisms. The vertical morphisms and special cells form a bicategory denoted by $\mathrm{Vert}(\mathbb{D})$.

Examples

$\mathbb{R}\mathrm{el}$: sets X, $\underset{\text{functions}}{X \longrightarrow Y}$, $\underset{\text{relations}}{X_s \longrightarrow\!\!\!\!\!\!\bullet\;\; X_t}$,
$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \subseteq & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_1 \end{array}$$

$\mathbb{C}\mathrm{at}$: categories X, $\underset{\text{functors}}{X \longrightarrow Y}$, $\underset{\text{profunctors}}{X_s \longrightarrow\!\!\!\!\!\!\bullet\;\; X_t}$,
$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \rightarrow & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_1 \end{array}$$

$\mathbb{P}\mathrm{os}$: posets X, $\underset{\text{monotone}}{X \longrightarrow Y}$, $\underset{\text{order ideals}}{X_s \longrightarrow\!\!\!\!\!\!\bullet\;\; X_t}$,
$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\big\downarrow\!\bullet & \subseteq & \bullet\!\big\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_1 \end{array}$$

$\mathbb{M}\mathrm{et}$: $\mathcal{V}$-$\mathbb{C}\mathrm{at}$ for $\mathcal{V} = [0, \infty]$, Lawvere metric spaces

Companions and Conjoints

A companion for $X \xrightarrow{f} Y$ is a vertical morphism $X \overset{f_*}{\longrightarrow\!\!\!\!\!\!\bullet\;\;\;} Y$ and cells

$$
\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
{\scriptstyle \mathrm{id}_X^\bullet}\big\downarrow & \eta & \big\downarrow{\scriptstyle f_*} \\
X & \xrightarrow[f]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle f_*}\big\downarrow & \varepsilon & \big\downarrow{\scriptstyle \mathrm{id}_Y^\bullet} \\
Y & \xrightarrow[\mathrm{id}_Y]{} & Y
\end{array}
$$

whose horizontal and vertical compositions are identities.

A conjoint for f is a vertical morphism $Y \overset{f^*}{\longrightarrow\!\!\!\!\!\!\bullet\;\;\;} X$ and cells

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \mathrm{id}_X^\bullet}\big\downarrow & \alpha & \big\downarrow{\scriptstyle f^*} \\
X & \xrightarrow[\mathrm{id}_X]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & \xrightarrow{\mathrm{id}_Y} & Y \\
{\scriptstyle f^*}\big\downarrow & \beta & \big\downarrow{\scriptstyle \mathrm{id}_Y^\bullet} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

Note: $\mathbb{R}\mathrm{el}, \mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}$, and $\mathbb{M}\mathrm{et}$ have all companions and conjoints.

Cauchy Completeness

Proposition

If f has a companion and conjoint, then $f_* \dashv f^*$ in $\mathrm{Vert}(\mathbb{D})$.

Definition

An object Y of $\mathbb{D}$ is Cauchy complete if every left adjoint vertical morphism $v\colon X \to\!\!\!\!\bullet\!\!\to Y$ is the companion of some $f\colon X \to Y$.

Exercise

Every set Y is Cauchy complete in $\mathbb{R}\mathrm{el}$.

Remark

Cauchy completeness was considered in the 70s and 80s for metric spaces, categories, and posets (see Borceux/Dejean).

Locales

Companions and conjoints played a role in a double category construction [N 2012] of exponentials of locally closed inclusions for locales, toposes, and topological spaces using Artin-Wraith glueing.

$$\mathbb{L}\mathrm{oc}\colon\ \text{locales } X, \quad \underset{\text{locale maps}}{X \xrightarrow{f} Y}, \quad \underset{\text{finite } \wedge\text{-maps}}{X_s \multimap X_t}, \quad \begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow\!\bullet & \geq & \bullet\!\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$$

Note: A "locale map" f has a finite $\wedge$-preserving left adjoint f^*.

Proposition

Every locale is Cauchy complete in $\mathbb{L}\mathrm{oc}$.

Proof.

Suppose $v\colon X \multimap Y$ is left adjoint to $w\colon Y \multimap X$ in $\mathbb{L}\mathrm{oc}$. Then $vw \geq \mathrm{id}^\bullet_Y$ and $\mathrm{id}^\bullet_X \geq wv$, and so $w \dashv v$ as poset maps. Since v preserves finite meets, it follows that v is a locale morphism such that $v_* = v$ in $\mathbb{L}\mathrm{oc}$. □

Toposes

$\mathbb{T}\mathrm{opos}$: toposes X, $\underset{\text{geom. morph.}}{X \longrightarrow Y}$, $\underset{\text{lex}}{X_s \longrightarrow\!\!\!\!\!\bullet\; X_t}$, $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow\bullet & \leftarrow & \bullet\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$

Note: A "geometric morphism" f has a left exact left adjoint f^*.

Proposition

Every topos is Cauchy complete in $\mathbb{T}\mathrm{opos}$.

Proof.

Suppose $v \dashv w$ in $\mathbb{T}\mathrm{opos}$. Then we have cells $vw \longleftarrow \mathrm{id}^{\bullet}_{Y}$ and $\mathrm{id}^{\bullet}_{X} \longleftarrow wv$, satisfying the adjunction identities, and so $w \dashv v$ as functors. Since v preserves finite limits, it follows that v is a geometric morphism such that $v_* = v$ in $\mathbb{T}\mathrm{opos}$. $\square$

Topological Spaces

$$\mathbb{T}\mathrm{op}\colon\ \text{top spaces } X,\quad \underset{\text{cont maps}}{X \longrightarrow Y},\quad \underset{\text{lex}}{\frac{X_s \longrightarrow\!\!\!\!\!\bullet\; X_t}{\mathcal{O}(X_s) \longrightarrow \mathcal{O}(X_t)}},\quad \begin{array}{ccc} \mathcal{O}(X_s) & \xrightarrow{\mathcal{O}(f_s)} & \mathcal{O}(Y_s) \\ {\scriptstyle v}\big\downarrow\!\bullet & \supseteq & \bullet\!\big\downarrow{\scriptstyle w} \\ \mathcal{O}(X_t) & \xrightarrow[\mathcal{O}(f_t)]{} & \mathcal{O}(Y_t) \end{array}$$

Recall [PTJ] a space Y is sober iff morphisms $f\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ of locales correspond bijectively to continuous maps $f\colon X \longrightarrow Y$.

Proposition

A space Y is Cauchy complete in $\mathbb{T}\mathrm{op}$ iff it is a sober space.

Proof.

Left adjoints $X \multimap Y$ in $\mathbb{T}\mathrm{op}$ are the left adjoints $\mathcal{O}(X) \multimap \mathcal{O}(Y)$ in $\mathbb{L}\mathrm{oc}$, and so Y is Cauchy complete in $\mathbb{T}\mathrm{op}$ iff it is sober. $\square$

Quantales

$\mathbb{Quant}$: quantales X, $X \xrightarrow[\text{morphisms}]{f} Y$, $X_s \xrightarrow[\text{lax}]{v\ \bullet} X_t$, $\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ {\scriptstyle v}\downarrow\bullet & \leq & \bullet\downarrow{\scriptstyle w} \\ X_t & \xrightarrow[f_t]{} & Y_t \end{array}$

Note: v is monotone with $v(x)v(x') \leq v(xx')$ and $e \leq v(e)$.

Proposition

Every quantale Y is Cauchy complete in $\mathbb{Quant}$.

Proof.

Suppose $v \dashv w$ in $\mathbb{Quant}$, where $X \xrightarrow{v\ \bullet} Y$. Since v is lax and preserves $\bigvee$, to see it is a quantale morphism, it suffices to show $v(e_X) \leq e_Y$ and $v(xx') \leq v(x)v(x')$. But, $e_X \leq w(e_Y)$ and

$$xx' \leq wv(x)wv(x') \leq w(v(x)v(x'))$$

Thus, Y is Cauchy complete in $\mathbb{Quant}$. $\square$

Adjoints in Double Categories

Suppose $\mathcal{V}$ is a bicomplete symmetric monoidal closed category, and consider the double category of monoids in $\mathcal{V}$

$$\mathbb{B}\mathrm{im}(\mathcal{V})\text{: monoids } R\ , \quad \underset{\text{homoms}}{R \xrightarrow{f} S}, \quad \underset{(R_t,\, R_s)\text{-bimods}}{R_s \xrightarrow{M} R_t}, \quad \begin{array}{ccc} R_s & \xrightarrow{f_s} & S_s \\ {\scriptstyle M}\downarrow & \to & \downarrow{\scriptstyle N} \\ R_t & \xrightarrow[f_t]{} & S_t \end{array}$$

Given an (S, R)-module M, there is a functor

$$M \otimes_R -\colon (R, Q)\text{-Mod} \longrightarrow (S, Q)\text{-Mod}$$

which has a right adjoint

$$S\mathrm{Mod}(M, -)\colon (S, Q)\text{-Mod} \longrightarrow (R, Q)\text{-Mod}$$

For commutative rings, $M \otimes_R -$ has a left adjoint iff M is fg projective as an R-module. Can we relate this to right adjoints to $M\colon R \longrightarrow S$? What about rigs/quantales? Non-commutative case?

Adjoints in Double Categories

Theorem

TFAE for $M \colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ *with* S*-presentation* $\sqcup_\alpha S \rightrightarrows \sqcup_\beta S \longrightarrow M$.

(a) $M \colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ *has a right adjoint in* $\mathbb{B}\mathrm{im}(\mathcal{V})$.

(b) $(Q, S)\text{-}\mathrm{Mod}(\mathcal{V}) \xrightarrow{-\otimes_S M} (Q, R)\text{-}\mathrm{Mod}(\mathcal{V})$ *has a left adjoint,* $\forall Q$.

(c) $(Q, S)\text{-}\mathrm{Mod}(\mathcal{V}) \xrightarrow{-\otimes_S M} (Q, R)\text{-}\mathrm{Mod}(\mathcal{V})$ *preserves limits.*

(d) $S\mathrm{Mod}(M, S) \otimes_S M \xrightarrow{\theta} S\mathrm{Mod}(M, M)$ *is invertible.*

Note: $(b) \Rightarrow (c) \Rightarrow (d)$ is like [NW]; can prove $(d) \Rightarrow (a) \Rightarrow (b)$.

Corollary

TFAE for an (S, R)*-module* M *over quantales (resp., rings, rigs).*

(a) $M \colon R \longrightarrow\!\!\!\!\!\!\bullet\;\; S$ *has a right adjoint in* $\mathbb{B}\mathrm{im}$.

(b) $- \otimes_S M$ *has a left adjoint.*

(c) M *is (resp., fg) projective as an* S*-module.*

Note: One can prove (c) iff θ is an invertible.

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