Free bicompletion of categories revisited (Part 1)

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Abstract

Whitman's theory of free lattices can be extended to lattices enriched over a quantale, to bicomplet categories, and also to bicomplete ∞-categories. It has applications to the semantic of linear logic [HJ1][HJ2].

My goal here is to introduce a few basic ideas of the theory of free bicomplete categories.

Apology

For 25 years, I have been promising to many people a draft of my paper on free bicompletion of categories. I apologise for been so late delivering. I am presently writing that draft, and I plan to finish it this Spring.

Plan

- Whitman's theory of free lattices
- Free bicomplete categories
- Atomic objects, soft categories
- Exact-coexact factorisations
- Rigid model structures

Whitman's theory of free lattices

A *lattice* is a poset L with binary infima (denoted $x \wedge y$ and binary suprema (denoted $x \vee y$). The notion of lattice is algebraic.

A lattice L has two operations, $\wedge, \vee : L \times L \to L$ and the following axioms hold:

- associativity:
  $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad x \vee (y \vee z) = (x \vee y) \vee z$
- commutativity:
  $x \wedge y = y \wedge x, \qquad x \vee y = y \vee x$
- idempotence:
  $x \wedge x = x, \qquad x \vee x = x$
- absorbtion:
  $x \wedge (x \vee y) = x \qquad x \vee (x \wedge y) = x$

Whitman's theory of free lattices

Let us denote by *Pos* the category of posets and order preserving maps, and by $\mathcal{L}at$ the category of lattices. Then the forgetful functor $\mathcal{L}at \to Pos$ has a left adjoint $\mathcal{L} : Pos \to \mathcal{L}at$ which takes a poset P to the free lattice $\mathcal{L}(P)$ generated by P. Let $i : P \to \mathcal{L}(P)$ be the canonical order preserving map.

Theorem

(Whitman) *For every* $u, v, x, y \in \mathcal{L}(P)$ *and* $a, b \in P$,

- *if* $x \wedge y \leq u \vee v$ *then*
  $x \wedge y \leq u$ *or* $x \wedge y \leq v$ *or* $x \leq u \vee v$ *or* $y \leq u \vee v$;
- *if* $i(a) \leq u \vee v$ *then* $i(a) \leq u$ *or* $i(a) \leq v$;
- *if* $x \wedge y \leq i(b)$ *then* $x \leq i(b)$ *or* $y \leq i(b)$;
- *if* $i(a) \leq i(b)$ *then* $a \leq b$.

Conversely, if L *is a lattice and* $i : P \to L$ *is an order preserving map satisfying the conditions above, and if* L *is generated by* $i(P)$, *then* $L = \mathcal{L}(P)$.

α-complete lattices

Let α-be a regular cardinal.

Definition

We say that a lattice L is *α-complete* if every subset $S \subseteq L$ of cardinality $< \alpha$ has a supremum $\bigvee S \in L$ and an infimum $\bigwedge S \in L$.

Let us denote by ${}^{\alpha}\mathcal{L}at$ the category of α-complete lattices. The forgetful functor ${}^{\alpha}\mathcal{L}at \to Pos$ has a left adjoint ${}^{\alpha}\mathcal{L} : Pos \to {}^{\alpha}\mathcal{L}at$ which takes a poset P to the α-complete lattice ${}^{\alpha}\mathcal{L}(P)$ freely generated by P.

Indecomposable elements

Let E be an α-complete lattice.

Definition

An element $a \in E$ is said to be α-*indecomposable* if the following conditions hold for every subset $S \subseteq E$ of cardinality $< \alpha$:

1. $a \leq \bigvee S \implies a \leq x$ for some $x \in S$;
2. $\bigwedge S \leq a \implies x \leq a$ for some $x \in S$.

Lemma

(Whitman) *The map $i : P \to {}^{\alpha}\mathcal{L}(P)$ induces an isomorphism between P and the poset of α-indecomposable elements of ${}^{\alpha}\mathcal{L}(P)$.*

Whitman's theory for α-complete lattices

Definition

We say that an α-complete lattice L is *α-soft*, if the following implication holds

$$\bigwedge S \leq \bigvee T \implies \begin{cases} s \leq \bigvee T & \text{for some } s \in S \\ \text{or} & \\ \bigwedge S \leq t & \text{for some } t \in T \end{cases} \tag{1}$$

for every pair of subsets $S, T \subseteq L$ of cardinality $< \alpha$.

Theorem

(Whitman) *An α-complete lattice L is free if an only if it is α-soft and generated by its α-indecomposable elements.*

Complete, cocomplete and bicomplete categories

Recall that a (locally small) category $\mathcal{C}$ is said to be *complete* (resp. cocomplete) if every diagram $D : I \to \mathcal{C}$ has a limit $\varprojlim D \in \mathcal{C}$ (resp. a colimit $\varinjlim D \in \mathcal{C}$). We say that a category $\mathcal{C}$ is *bicomplete* if it is complete and cocomplete

Recall that a functor between complete (resp. cocomplete) categories $F : \mathcal{C} \to \mathcal{D}$ is said to be *continuous* (resp. cocontinuous) if it preserves limits (resp. colimits). We say that a functor between bicomplete categories is *bicontinuous* if it continuous and cocontinuous.

Free completion, cocompletion and bicompletion

Every locally small category $\mathcal{K}$ admits a locally small

- free cocompletion $\sigma : \mathcal{K} \to \Sigma(\mathcal{K})$
- free completion $\pi : \mathcal{K} \to \Pi(\mathcal{K})$
- free bicompletion $\lambda : \mathcal{K} \to \Lambda(\mathcal{K})$

It is far from obvious that $\Lambda(\mathcal{K})$ is locally small when $\mathcal{K}$ is locally small.

The cocompletion $\Sigma(\mathcal{K})$

The category $\Sigma(\mathcal{K})$ is cocomplete and the functor

$$\sigma^\star : Fun^{cc}(\Sigma(\mathcal{K}), \mathcal{E}) \to Fun(\mathcal{K}, \mathcal{E})$$

is an equivalence of categories for any cocomplete category $\mathcal{E}$.

When $\mathcal{K}$ is small, $\Sigma(\mathcal{K})$ is the presheaf category $Psh(\mathcal{K}) = Fun(\mathcal{K}^{op}, Set)$

When $\mathcal{K}$ is locally small, $\Sigma(\mathcal{K})$ is the category of presentable presheaves $\mathcal{K}^{op} \to Set$.

By definition, a presheaf $F : \mathcal{K}^{op} \to Set$ is *presentable* if it it the colimit

$$F = \varinjlim_{i \in I} Hom(-, A(i))$$

of a diagram of representables $A : I \to \mathcal{K}$.

σ-atomic objects

We say that an object A in a cocomplete category $\mathcal{C}$ is σ-*atomic* if the functor

$$\mathcal{C}(A, -) : \mathcal{C} \to Set$$

is cocontinuous.

A retract of a σ-atomic object is σ-atomic.

If $\sigma : \mathcal{K} \to \Sigma(\mathcal{K})$, then an object $A \in \Sigma(\mathcal{K})$ is σ-atomic if and only if it is a retract of an object $\sigma(K)$ for some $K \in \mathcal{K}$.

Theorem

A cocomplete category $\mathcal{C}$ is free if and only it is generated (under colimits) by σ-atomic objects.

The free completion $\pi : \mathcal{K} \to \Pi(\mathcal{K})$

The category $\Pi(\mathcal{K})$ is complete and the functor

$$\pi^\star : \mathit{Fun}^c(\Pi(\mathcal{K}), \mathcal{E}) \to \mathit{Fun}(\mathcal{K}, \mathcal{E})$$

is an equivalence of categories for any complete category $\mathcal{E}$.

The category $\Pi(\mathcal{K})$ is the opposite of the category $\Sigma(\mathcal{K}^{op})$, and the functor $\pi : \mathcal{K} \to \Pi(\mathcal{K})$ is the opposite of the functor $\sigma : \mathcal{K}^{op} \to \Sigma(\mathcal{K}^{op})$.

When $\mathcal{K}$ is small, $\Pi(\mathcal{K}) = \mathit{Fun}(\mathcal{K}, \mathit{Set})^{op}$ and the functor π is the opposite of the Yoneda functor $y : \mathcal{K}^{op} \to \mathit{Fun}(\mathcal{K}, \mathit{Set})$.

π-atomic objects

We say that an object A in a complete category $\mathcal{C}$ is π-*atomic* if the functor

$$\mathcal{C}(-, A) : \mathcal{C}^{op} \to Set$$

is cocontinuous.

An object $A \in \mathcal{C}$ is π-atomic if and only if the opposite object $A^{op} \in \mathcal{C}^{op}$ is σ-atomic.

A retract of a π-atomic object is π-atomic.

If $\pi : \mathcal{K} \to \Pi(\mathcal{K})$, then object $A \in \Pi\mathcal{K}$ is π-atomic if and only if it is a retract of an object $\pi(K)$ for some $K \in \mathcal{K}$.

Theorem

A complete category $\mathcal{C}$ is free if and only it is generated (under limits) by π-atomic objects.

Side remarks on completely distributive categories

Completely distributive categories are bicomplete but not free (as bicomplete categories).

Lemma

[Day-Lack] *The category $\Sigma\mathcal{C}$ is complete if $\mathcal{C}$ is complete.*

We say that a bicomplete category $\mathcal{C}$ is *completely distributive* if the colimit functor $\varinjlim : \Sigma\mathcal{C} \to \mathcal{C}$ is continuous.

Let $\mu : \mathcal{K} \to \Sigma\Pi(\mathcal{K})$ be the composite

$$\mathcal{K} \xrightarrow{\sigma} \Sigma\mathcal{K} \xrightarrow{\Sigma(\pi)} \Sigma\Pi\mathcal{K}$$

Theorem

[Marmolejo, Rosebrugh, Wood] *The functor $\mu : \mathcal{K} \to \Sigma\Pi(\mathcal{K})$ exhibits the completely distributive category freely generated by $\mathcal{K}$.*

The free bicompletion $\lambda : \mathcal{K} \to \Lambda\mathcal{K}$

The category $\Lambda(\mathcal{K})$ is bicomplete and the functor

$$\lambda^\star : Fun^{bc}(\Lambda(\mathcal{K}), \mathcal{E}) \to Fun(\mathcal{K}, \mathcal{E})$$

is an equivalence of categories for any bicomplete category $\mathcal{E}$.

We say that an object in a bicomplete category $\mathcal{C}$ is *atomic* if it is both σ- and π-atomic.

If $\lambda : \mathcal{K} \to \Lambda(\mathcal{K})$, then an object $A \in \Lambda(\mathcal{K})$ is atomic if and only if it is a retract of an object $\lambda(K)$ for some $K \in \mathcal{K}$.

Theorem

A bicomplete category $\mathcal{C}$ is free if and only it is **soft** *and generated (under limits and colimits) by atomic objects.*

We next define the notion of soft category.

Soft categories

Definition

If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are cocomplete categories, we say that a functor of two variables $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is *soft* if the following square of canonical maps

$$\begin{array}{ccc} \varinjlim \varinjlim F(A, B) & \longrightarrow & \varinjlim F(A, \varinjlim B) \\ \downarrow & & \downarrow \\ \varinjlim F(\varinjlim A, B) & \longrightarrow & F(\varinjlim A, \varinjlim B). \end{array} \tag{2}$$

is a pushout for every pair of diagrams $A : I \to \mathcal{C}$ and $B : J \to \mathcal{D}$.

Soft categories

Definition

We say that a bicomplete category $\mathcal{C}$ is *soft* if the functor

$$Hom : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set} \tag{3}$$

is soft.

By definition, $\mathcal{C}$ is soft if the following square of canonical maps

$$\begin{array}{ccc} \varinjlim \varinjlim Hom(A, B) & \longrightarrow & \varinjlim Hom(A, \varinjlim B) \\ \downarrow & & \downarrow \\ \varinjlim Hom(\varprojlim A, B) & \longrightarrow & Hom(\varprojlim A, \varinjlim B). \end{array} \tag{4}$$

is a pushout for every pair of diagrams $A : I \to \mathcal{C}$ and $B : J \to \mathcal{C}$.

Exact natural transformations

Definition

If $\mathcal{C}$ and $\mathcal{D}$ are complete categories, we say that a natural transformation $u : F \to G : \mathcal{C} \to \mathcal{D}$ is **exact** if the following square of canonical maps is a pullback,

$$\begin{array}{ccc} F(\varprojlim A) & \longrightarrow & \varprojlim FA \\ {\scriptstyle u(\varprojlim A)}\downarrow & & \downarrow{\scriptstyle \varprojlim uA} \\ G(\varprojlim A) & \longrightarrow & \varprojlim GA. \end{array}$$

for any diagram $A : I \to \mathcal{C}$.

Remark: If $\top$ is the terminal functor $\mathcal{C} \to \mathcal{D}$, then the natural transformation $F \to \top$ is exact iff the functor F is continuous.

Coexact natural transformations

Definition

If $\mathcal{C}$ and $\mathcal{D}$ are cocomplete categories, we say that a natural transformation $u : F \to G : \mathcal{C} \to \mathcal{D}$ is **coexact** if the following square of canonical maps is a pushout

$$\begin{array}{ccc} \varinjlim FA & \longrightarrow & F(\varinjlim A) \\ {\scriptstyle \varinjlim uA}\downarrow & & \downarrow{\scriptstyle u(\varinjlim A)} \\ \varinjlim GA & \longrightarrow & G(\varinjlim A). \end{array}$$

for any diagram $A : I \to \mathcal{C}$.

Remark: If $\perp$ is the initial functor $\mathcal{C} \to \mathcal{D}$, then the natural transformation $\perp \to G$ is coexact iff the functor G is cocontinuous.

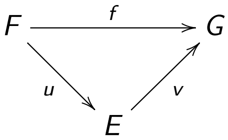
Two factorisations

Let $\lambda : \mathcal{K} \to \Lambda(\mathcal{K})$ the free bicompletion of a category $\mathcal{K}$.

If $\mathcal{S}$ is a category, we say that a natural transformation $f : F \to G : \Lambda(\mathcal{K}) \to \mathcal{S}$ is a λ-*equivalence* if the natural transformation $\lambda^*(f) = f \circ \lambda : F \circ \lambda \to G \circ \lambda$ is invertible.

Lemma

If the category $\mathcal{S}$ is bicomplete, then every natural transformation $f : F \to G : \Lambda(\mathcal{K}) \to \mathcal{S}$ admits a unique factorisation



with $u : F \to E$ a coexact λ-equivalence and $v : E \to G$ an exact transformation. There is a dual factorisation with u a coexact transformation and v an exact λ-equivalence.

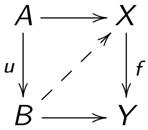
Factorisation systems

Definition

A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is called a *factorisation system* if the following conditions hold:

- the classes $\mathcal{A}$ and $\mathcal{B}$ contain the isomorphisms and are closed under composition;
- every map $f : A \to B$ admits a unique factorisation $f = vu : A \to E \to B$ with $u \in \mathcal{A}$ and $v \in \mathcal{B}$ (the factorisation is unique up to unique iso).

It follows from these conditions that if $u \in \mathcal{A}$ and $f \in \mathcal{B}$, then every commutative square



has a unique diagonal filler $B \to X$.

Rigid model structures

Definition

Let $\mathcal{E}$ be a category with finite limits and finite colimits. A *rigid model structure* on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ satisfying the following conditions:

1. the class $\mathcal{W}$ contains the isomorphisms and has the 3-for-2 property;
2. the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and the pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are *factorisation systems*.

A map in $\mathcal{W}$ is said to be a *weak-equivalence*.

A map in $\mathcal{F}$ is said to be a *fibration*. An object $X \in \mathcal{E}$ is said to be *fibrant* if the map $X \to \top$ is a fibration. A map in $\mathcal{F} \cap \mathcal{W}$ is said to be a *trivial fibration*.

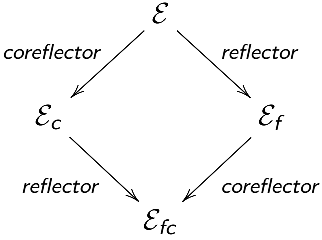
A map in $\mathcal{C}$ is said to be a *cofibration*. An object $X \in \mathcal{E}$ is said to be *cofibrant* if the map $\bot \to X$ is a cofibration. A map in $\mathcal{C} \cap \mathcal{W}$ is said to be a *trivial cofibration*.

The homotopy category of a rigid model category

The subcategory $\mathcal{E}_f$ of fibrant objects (resp. $\mathcal{E}_c$ of cofibrant objects) of a rigid model model category $\mathcal{E}$ is reflective (resp. coreflective).

The intersection $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$ is coreflective in $\mathcal{E}_f$ and reflective in $\mathcal{E}_c$.

Moreover, the following square commutes:



A rigid model structures on $Fun(\Lambda(\mathcal{K}), \mathcal{S})$

Let $\lambda : \mathcal{K} \to \Lambda(\mathcal{K})$ be the bicompletion of a category $\mathcal{K}$.

Theorem

If $\mathcal{S}$ is a bicomplete category, then the category $Fun(\Lambda(\mathcal{K}), \mathcal{S})$ admits a rigid model structure in which a weak equivalence is an λ-equivalence, a fibration is an exact natural transformation and a cofibration is a coexact natural transformation.

A fibrant (resp. cofibrant) object is a continuous (resp. cocontinuous) functor $\Lambda(\mathcal{K}) \to \mathcal{S}$

A fibrant-cofibrant object is a bicontinuous functor $\Lambda(\mathcal{K}) \to \mathcal{S}$

The category of bicontinuous functor $\Lambda(\mathcal{K}) \to \mathcal{S}$ is equivalent to the category $Fun(\mathcal{K}, \mathcal{S})$

Fibrant objects in a rigid model structure

Let $\mathcal{E}$ be a category equipped with a rigid model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

The *fibrant replacement* $A \to A_f$ of an object $A \in \mathcal{E}$ is obtained by factoring the map $A \to \top$ as a trivial cofibration $A \to A_f$ followed by a fibration $A_f \to \top$.

A map $r : A \to B$ is reflecting the object A into $\mathcal{E}_f$ if and only if the following two conditions hold:

1. B is fibrant
2. r is a trivial cofibration.

Best continuous approximation

Let $\lambda : \mathcal{K} \to \Lambda(\mathcal{K})$ the bicompletion of a category $\mathcal{K}$.

Corollary

The subcategory $Fun^c(\Lambda(\mathcal{K}), \mathcal{S})$ *of continuous functors* $\Lambda(\mathcal{K}) \to \mathcal{S}$ *is reflective.*

For every functor $F : \Lambda(\mathcal{K}) \to \mathcal{S}$ there exists a best approximation $r : F \to F^c$ by a continuous functor $F^c : \Lambda(\mathcal{K}) \to \mathcal{S}$.

Corollary

The natural transformation $r : F \to F^c$ *is a coexact* λ*-equivalence.*

An example

For any diagram $A : I \to \Lambda(\mathcal{K})$, the map

$$\varinjlim Hom(A, X) \to Hom(\varprojlim A, X)$$

is a natural transformation $r(X) : F(X) \to Hom(L, X)$, where $F(X) = \varinjlim Hom(A, X)$ and $L = \varprojlim A$.

Lemma

The natural transformation $r : F \to Hom(L, -)$ is coexact.

Proof.

It suffices to show that $r : F \to Hom(L, -)$ exhibits the best approximation of F by a continuous functor. Let us show that the map $Nat(r, G) : Nat(Hom(L, -), G) \to Nat(F, G)$ is invertible for every continuous functor $G : \Lambda(\mathcal{K}) \to \mathrm{Set}$. We have

$$\begin{aligned} Nat(F, G) &= \varprojlim Nat(Hom(A, -), G) = \varprojlim GA && (5) \\ &= G(\varprojlim A) = G(L) = Nat(Hom(L, -), G) && (6) \end{aligned}$$

since the functor G is continuous. □

$\Lambda(\mathcal{K})$ is soft

We saw that the natural transformation

$$\varinjlim Hom(A, X) \to Hom(\varprojlim A, X)$$

is coexact for any diagram $A : I \to \Lambda(\mathcal{K})$. Hence the following square is a pushout

$$\begin{array}{ccc} \varinjlim \varinjlim Hom(A, B) & \longrightarrow & \varinjlim Hom(A, \varinjlim B) \\ \downarrow & & \downarrow \\ \varinjlim Hom(\varprojlim A, B) & \longrightarrow & Hom(\varprojlim A, \varinjlim B). \end{array} \quad (7)$$

for every diagram $B : J \to \Lambda(\mathcal{K})$.

Conclusions

We saw that Whitman's theory of free lattices can be extended to free bicomplete categories. It can also be extended to

- free bicomplete enriched categories,
- free bicomplete ∞-categories,
- free bicomplete enriched ∞-categories.

and the proof are essentially the same. The theory can also be extended to categories that are simultaneously closed under a class α of limits and a class β of colimits ([AK][KP][ABLR][KS][LG][Rezk 1,2]). The (α, β)-bicompletion

$$\lambda : \mathcal{K} \to \Lambda^{(\alpha,\beta)}(\mathcal{K})$$

of a category $\mathcal{K}$ is (α, β)-soft.

The theory of bicompletion appears to be a fundamental aspect of general category theory.

Applications to linear logic

Free bicomplete lattices have a game theoretic interpretation related to Lorenzen's game theoretic interpretation of logic [Bla] [J3]. The category of coherence spaces of Girard is pointed and soft with respect to product and coproducts [HJ1] [HJ2]; it can be used to construct free pointed category with products and coproducts. The category $\Lambda(1)$ is star-autonomous, but an explicit combinatorial construction is still missing.

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Thank you for your attention!