The Fundamental Theorem of Calculus, point-free

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Topos Institute, 30 May 2024

1. Point-free topology

   What? Subtle topos theory ...

   How? ... but unobtrusively.

   Why? Generalized spaces; bundles; fresh insights

2. Example: Fundamental Theorem of Calculus

Point-free topology: What?

Topology point-*set*

Space = set X of points + topology $\tau \subseteq \mathcal{P}X$ (*open* subsets)

Topology point-*free*

Space = logical theory describing points
Point = model of theory

Theory = Signature

Set of propositional symbols
– generates *formulae* (using $\wedge$, $\bigvee$)
– they correspond to opens

+ axioms

formula $\vdash$ formula
If left hand formula is true in a model, then so is right hand.

Model –

- Assign truth value to each propositional symbol.
- Extends to all formulae.
- For each axiom $\phi \vdash \psi$, if ϕ true then require ψ true.

Example: real line $\mathbb{R}$

Signature

For each rational $q \in \mathbb{Q}$: two propositional symbols $[\cdot < q]$ and $[q < \cdot]$.

Axioms – eg

$$
\begin{aligned}
[\cdot < q] &\vdash \bigvee_{q' < q} [\cdot < q'] \\
[\cdot < q] \wedge [q < \cdot] &\vdash \bot \\
\top &\vdash [q < \cdot] \vee [\cdot < r] \quad (\text{if } q < r)
\end{aligned}
$$

Model:

Specify which rationals are bigger, which are smaller.

That's a real number described as a *Dedekind cut*.

Point-free topology: How?

Think of maps $f : \mathbb{R} \to \mathbb{R}$ in a style of programming languages.

Let x:$\mathbb{R}$
$\vdots$
$f(x)$:$\mathbb{R} := \cdots$
$\vdots$

Declare formal parameter x.
Do some auxiliary calulations.
Define result $f(x)$ as model:

- specify truth values $[f(x) < q]$, $[q < f(x)]$,
- prove that axioms hold.

eg absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}$

Let x:$\mathbb{R}$

$$[|x| < q] := [x < q] \wedge [-q < x]$$
$$[q < |x|] := [q < x] \vee [x < -q]$$

... and prove axioms

Inside the box, in scope of x, is a *different mathematics!*

1. Lots of non-standard truth values

$[x < q]$, $[q < x]$ (for each rational q)

They don't express *whether* something is true, rather *where* (ie for which models x) it is true.

2. Continuity = different logic

Continuity: inverse image of open is open

$f^{-1}([x < q]) = [f(x) < q]$ is made from truth values of form $[x < r]$ and $[s < x]$ using $\wedge$ and $\bigvee$.
Similarly for $[q < f(x)]$.

We want continuity, therefore restrict mathematics inside box to limit how we construct $f(x)$.

Geometric mathematics

Mathematics of *sets* highly restricted

- finite limits
- colimits
- includes natural numbers $\mathbb{N}$, $\mathbb{Q}$, free algebras

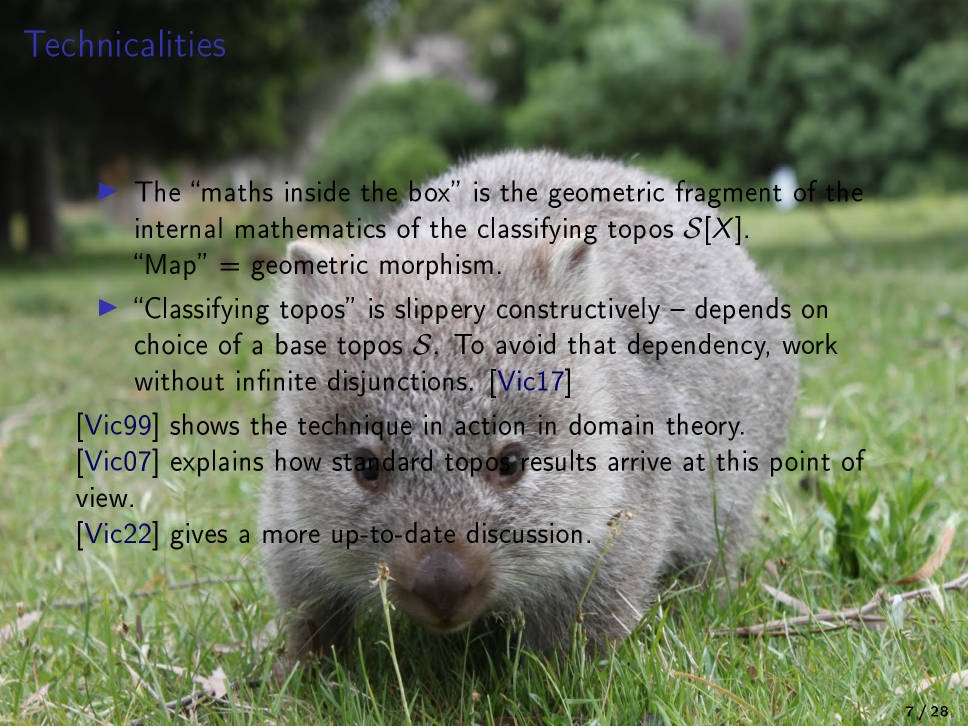
Function spaces Y^X, powersets $\mathcal{P}X$, the real line $\mathbb{R}$ are *not sets!*
They must be dealt with as (point-free) spaces.

Corresponding logic:

$\wedge$, $\bigvee$, $=$, $\exists$. Not $\neg$, $\rightarrow$, $\forall$, except implicitly in logical axioms:
$\phi \vdash_q \psi$ meaning $\forall q(\phi(q) \rightarrow \psi(q))$.

Infinite $\bigvee$ can often be avoided by using $\exists$ with an infinite set. eg

$$[\cdot < q] \vdash_{q:\mathbb{Q}} (\exists q'{:}\mathbb{Q})(q' < q \wedge [\cdot < q']) \text{ instead of } [\cdot < q] \vdash \bigvee_{q' < q} [\cdot < q'].$$



Technicalities

- The "maths inside the box" is the geometric fragment of the internal mathematics of the classifying topos $\mathcal{S}[X]$.
  "Map" = geometric morphism.
- "Classifying topos" is slippery constructively – depends on choice of a base topos $\mathcal{S}$. To avoid that dependency, work without infinite disjunctions. [Vic17]

[Vic99] shows the technique in action in domain theory.
[Vic07] explains how standard topos results arrive at this point of view.
[Vic22] gives a more up-to-date discussion.

Point-free topology: Why?

I: Generalized spaces

Can use *first-order* geometric theories to define spaces more general than those in point-set topology.

Then many proper classes can be expressed as point-free spaces: eg space of sets, space of groups

Point-free topology: Why?

II: Bundles as continuously indexed families of spaces

No natural way to do this point-set!

Using I: define spaces of presentations of geometric theories
– eg space of sites.
Presentation itself defines a space.
Suppose you do definition in the box for $x{:}X$. Can see it two ways.

As bundle

Map from X to a "space of spaces".
Each $x{:}X$ maps to the *fibre* over x.

As forgetful map to X

Use presentation to extend theory of X. Models are pairs (x, y) with $x{:}X$ and y in fibre.
Extended theory gives its own space, with map to X that forgets y.

[SVW14] uses geometric techniques to study some bundles appearing in quantum structure.

Point-free real analysis

Typical techniques

- One-sided reals: half of a Dedekind section
- Hyperspaces: spaces of subspaces
  Also, analogous spaces of measures.

Example: Fundamental Theorem of Calculus (FTC)

Need to deal with both –

- Integration: lower and upper integrals [Vic08]; see also[CS09].
- Differentiation: in a Carathéodory style, using existence of continuous slope maps. [Vic09] proves Rolle's Theorem.

Real-valued maps (including work with Ming Ng)

- Exponentiation and logarithms [NV22].
- FTC, applied to calculus of exp and log [Vic23].

One-sided reals

Topologies of semicontinuity ...

... separate out opens $[q < \cdot]$ (lower semicontinuity) and $[\cdot < q]$ (upper).

Write $\overrightarrow{\mathbb{R}}$ and $\overleftarrow{\mathbb{R}}$ for the spaces of lower and upper reals.
Note: $\overrightarrow{\mathbb{R}}$ includes ∞, $\overleftarrow{\mathbb{R}}$ includes $-\infty$

Then a Dedekind real x is a pair $(\underline{x}, \overline{x})\colon \overrightarrow{\mathbb{R}} \times \overleftarrow{\mathbb{R}}$, satisfying axioms

$$[\overline{x} < q] \wedge [q < \underline{x}] \vdash \bot$$
$$\top \vdash [q < \underline{x}] \vee [\overline{x} < r] \quad (\text{if } q < r)$$

$\underline{x}$ and $\overline{x}$ show how x is approximated by rationals from below and from above.

A useful strategy for real analysis

1. Deal separately with lower and upper parts – they are simpler in geometric mathematics.
2. Show that they fit together by proving the two axioms.

Limitation: All maps $\overrightarrow{\mathbb{R}} \to \overrightarrow{\mathbb{R}}$ must be monotone

Simple consequence of continuity. Similarly for $\overleftarrow{\mathbb{R}}$.
Hence,

- No subtraction $x - y$ – antitone in y.
- Can't multiply xy unless both non-negative.

Works fine for exponentiation and logs [NV22]

More generally – often need to separate out signed parts by more combinatorial means (example: integration).

Hyperspace:[1] space of (some) subspaces of another space

eg Vietoris hyperspace VX, a space of certain compact subspaces of X. Vietoris topology already known point-set.
Making this work point-free is not straightforward [Vic04], but once in place it can be intuitive to use.

eg Heine-Borel Theorem

... expressed as map $\mathrm{HB}_C(x, y){:}V\mathbb{R}$ for reals $x \leq y$ [Vic09].
This shows that the closed interval $[x, y]$ (is compact and) depends continuously on x and y.

Tautologous bundle

Each point of VX is a space, so get a bundle over VX.
Each point of the bundle space is a pair (K, x) where K is a (certain kind of) compact subspace of X, and x is in K.

[1] Point-free hyperspaces have historically been called powerlocales

Integration

Lower/upper integrals are approximated from below/above: suggests using lower/upper reals first, then glue together.

Lower integrals $\underline{\int}_X fd\mu$

f, μ both valued as lower reals.
Multiplying: hence must both be non-negative.
$f : X \to \overrightarrow{[0,\infty]}$.

μ is a *valuation* on X – like a measure defined on opens

μU:$\overrightarrow{[0,\infty]}$, $\mu(\emptyset) = 0$, $\mu U + \mu V = \mu(U \wedge V) + \mu(U \vee V)$
Continuous (preserves directed joins) –

$$\mu(\bigvee_{i\in I} U_i) = \sup_{I_0 \subseteq_{\mathrm{fin}} I} \mu(\bigvee_{i\in I_0} U_i)$$

Integration – one-sided results from [Vic08]

Evaluating lower integral $\underline{\int}_X f d\mu$

Not completely obvious – more like Choquet integral than Lebesgue.

Upper integrals $\overline{\int}_X f d\nu$

$f : X \to \overleftarrow{[0, \infty)}$
ν a *co*valuation – νU is measure of closed complement $X - U$.

Valuation space $\mathfrak{V}X$

= space of valuations on X
– analogous to Giry monad for measurable spaces.
Similarly $\mathfrak{C}X$ for covaluations.

Integration – 2-sided results from [Vic23]

$\int_X fd\mu$, with $f\colon X \to [0,\infty)$

μ still a valuation, but must be "finite" – μX is Dedekind.
Then it has a complementary covaluation $\neg\mu$, $(\neg\mu)U = \mu X - \mu U$.

Theorem – *Hardest calculation in paper!*

If X is compact, then $\underline{\int}_X fd\mu$ and $\overline{\int}_X fd(\neg\mu)$ together make a Dedekind real.
Thus we have defined $\int_X fd\mu{:}\mathbb{R}$.

Signed $f\colon X \to \mathbb{R}$

Split f as $f_+ - f_-$, where $f_+ = \max(f,0)$ etc., then define

$$\int_X fd\mu = \int_X f_+ d\mu - \int_X f_- d\mu$$

Riemann integrals $\int_x^y f(t)dt$

1. Define Lebesgue valuation $\lambda{:}\mathfrak{V}\mathbb{R}$.
   Suffices to define $\lambda(q, r) = r - q$ for rationals $q < r$, and check some conditions.
2. If $x \leq y$ it induces $\lambda_{xy}{:}\mathfrak{V}([x, y])$.
   Can now define $\int_x^y f(t)dt = \int_{[x,y]} fd\lambda_{xy}$.
3. For general x, y, define $\int_x^y f(t)dt = \pm \int_{\min(x,y)}^{\max(x,y)} f(t)dt$, where $\pm$ depends on order of x, y.
   Care needed for this geometrically! Can't just do a classical case split.
4. Prove
   $$\int_x^z f(t)dt = \int_x^y f(t)dt + \int_y^z f(t)dt.$$

That sets us up for integration in FTC.

Differentiation à la Carathéodory

$f : \mathbb{R} \to \mathbb{R}$ differentiable if –
– there is a (continuous) *slope* map $f^{\langle 1 \rangle} : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$f(y) - f(x) = (y - x) f^{\langle 1 \rangle}(x, y).$$

Then derivative f' defined by

$$f'(x) = f^{\langle 1 \rangle}(x, x).$$

Question geometrically: How do you define slope maps?
FTC: integrals are really useful.

FTC

Let f be defined on some real open interval by

$$f(x) = \int_{x_0}^{x} g(t)dt.$$

Then f is differentiable, with derivative $f' = g$.

What is slope map?

If $x < y$ (and $y < x$ easily reduced to this), then

$$f^{\langle 1 \rangle}(x, y) = \frac{\int_x^y g(t)dt}{y - x} = \int_{[x,y]} gd\frac{\lambda_{xy}}{y - x} = \int_{[x,y]} gd\upsilon_{xy},$$

where υ_{xy} is the *uniform probability* valuation (total mass $= 1$) on $[x, y]$.

Uniform probability valuation υ_{xy}

... makes sense even when $x = y$, so $[x, y] \cong 1$.

Then $\int_{[x,x]} g d\upsilon_{xx} = g(x)$.

Geometrically –

- Can define υ_{xy} in a way that covers all cases $x \leq y$.
- Geometricity ensures that everything varies continuously with x and y. That includes domain of integration $[x, y]$ and valuation υ_{xy}.
- Hence $f^{\langle 1 \rangle}$ varies continuusly with x, y even at $x = y$.

This completes the proof of FTC

Note how existence of integrals was used to give an explicit slope map.

Converse of FTC

If f is differentiable, then

$$\int_x^y f'(t)dt = f(y) - f(x).$$

Proof (more or less standard)

1. Fixing x_0, define $g_{x_0}(x) = \int_{x_0}^x f'(t)dt$.
2. By FTC, $g'_{x_0} = f'$, so $(f - g_{x_0})' = 0$.
3. Using Rolle's Theorem [Vic09], $f - g_{x_0}$ is constant, $f(x_0)$.
   QED

Application: $\log_\gamma$ is differentiable

Proof

1. Define
$$\ln x = \int_1^x \frac{dt}{t}.$$
2. ln is a homomorphism from $((0,\infty),\cdot)$ to $(\mathbb{R},+)$.
3. Lemma: if f is such a homomorphism then $f(\gamma^y) = yf(\gamma)$. Use algebraic laws to show for y rational, then density of rationals in reals.
4. Hence $f(x) = (\log_\gamma x)f(\gamma)$.
5. For ln, get $\log_\gamma x = (\ln x)/(\ln \gamma)$, an integral: now use FTC.

Corollary: Using chain rule, γ^y is differentiable in y.

Slope maps are defined using integrals! Don't have to find independent definition of γ^y/y as map, continuous at 0.

Point-free topology – Why? Fresh insights

Ming Ng [Ng22, NV] and number theory

He studied *Ostrowski's Theorem*: number-theoretic result concerning *absolute values* on $\mathbb{Q}$, maps $\mathbb{Q} \to \mathbb{R}$ analogous to ordinary $|\cdot|$.
There's an extra family of *p-adic* absolute values, one for each prime p.

Geometric reasoning invites us to consider using one-sided reals.

With upper reals, this automatically brings fresh insight by including "multiplicative seminorms", with unexpected connections to the Berkovich spectrum.

Point-free topology – Why? Fresh insights

Vickers: One-sided Minkowski space-time

Usual topology on space-time $\mathbb{R}^4$ can be split into two parts, analogous to lower and upper for reals.

In each one, the specialization order of the topology matches the causal order of physics (or its opposite).

You also find infinite points arising, just as (eg) lower reals have to include $+\infty$. It turns out that they match the ideal points already identified by Geroch-Kronheimer-Penrose [GKP72].

I'm exploring these spaces to see how much else of their intrinsic structure has physical significance.

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