Structure and Dynamics of Working Language

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TOPOS INSTITUTE

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Outline

Working language for three funders

This is the first year of a grant that's funded by three AFOSR programs:

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What is *working language*? How does language *work*?

- Language works in the sense of basic physics: it directs energy.
- If I say "pass the salt," 10^{25} atoms move through space.
- This involves compositional planning and high-precision control.
- Getting it set up in the first place requires (evolution and) learning.
- DNA is working language: ACGT symbols code for chemistry.
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Wanted: a minimally-assumptive mathematical framework to set up WL.

- Framing all this in strong/weak/gravity/EM forces, where's language?
- Want the structure and dynamics of language to be front and center.
- Can we relate matter and pattern using math rather than physics?

$\mathbb{C}$-like miracles in Poly

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- When your eyes are open vs. closed, the input datatype is different.
- Communication channels can change based on what is said over them.
- The category **Poly** fit very well, covering all the examples.

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- A "miracle" of $\mathbb{C}$: add roots of $x^2 + 1$, get differentiable $\implies$ analytic.
- A "miracle" of **Poly**: its comonoids are precisely categories.
- The work I'd done a decade earlier on data migration was subsumed.
- Automata, dependent types, dynamical systems, PL, all foregrounded.

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I became optimistic that **Poly** could be a unified framework for my research.

- It has potential to reveal deep insights about the nature of comput'n.
- Perhaps it would foreground something about how language works?
- Today: the way terminating scripts run on persistent machines...
- ...is foregrounded via free monads $\mathfrak{m}_p$ and cofree comonads $\mathfrak{c}_q$.

Plan

The plan for the rest of my talk is as follows:

- Being year 1, a review of basic category theory
- A review of polynomial functors
- Discuss new result: a module structure $\mathfrak{m}_p \otimes \mathfrak{c}_q \to \mathfrak{m}_{p \otimes q}$

This is joint work with Sophie Libkind.

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Categories, functors, and natural transformations

The big 3 of category theory are: category, functor, natural transformation.

- Category = relational fabric. Functor = mapping. NT=trajectory.
- Example: $(\mathbb{N}, \leq)$, $(- \times 2)\colon (\mathbb{N}, \leq) \to (\mathbb{N}, \leq)$, $(- \times 2) \leq (- \times 3)$.

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A category $\mathcal{C}$ is a relational "fabric", like a space.

- It's got a set $\mathrm{Ob}(\mathcal{C})$, *objects* $\approx$ "points".
- For each $c, c' \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Mor}(c, c')$, *morphisms* $f\colon c \to c'$.
- Identities id_c and compositions $f \mathbin{⨟} g$ that are unital and assoc.

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A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories is a "non-tearing" map.

- It sends each $c \in \mathrm{Ob}(\mathcal{C})$ to some $F(c) \in \mathrm{Ob}(\mathcal{D})$.
- It sends each morphism $f\colon c \to c'$ to some $F(f)\colon F(c) \to F(c')$.
- It preserves identities and composition, i.e. $F(f_1 \mathbin{;} f_2) = F(f_1) \mathbin{;} F(f_2)$.

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A natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$ is a $\mathcal{D}$-trajectory param'd by $\mathcal{C}$.

- For each $c \in \mathrm{Ob}(\mathcal{C})$, a morphism $\alpha_c \colon F(c) \to G(c)$ in $\mathcal{D}$.
- For each $f \colon c \to c'$ in $\mathcal{C}$, an equation $\alpha_c \mathbin{;} F(f) = F(f) \mathbin{;} \alpha_{c'}$.

Set, its endofunctors, and its monoidal structures

The big 4 would be the above, plus **Set**, the category of sets.

- The objects of **Set** are sets (in some mathematical universe of sets).
- A morphism $f\colon S \to T$ is a function, and composition is as usual.
- For any $N \in \mathbb{N}$, let $N := \{`1`, \ldots, `N`\}$ denote a set with N elements.
- Disjoint union $A + B$, Cartesian product $A \times B$, and exponential B^A.

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Let's think about functors $F\colon \mathbf{Set} \to \mathbf{Set}$.

- F needs to send each set to a set and function to a function.
- How about $X \mapsto X^2$? Or $X \mapsto X + 1$? Or $X \mapsto 7$? Or $X \mapsto 2^X$?
- Functors F, G can be added or multiplied, pointwise: $F + G$, $F \times G$.

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A *monoidal structure* $(I, \odot)$ on $\mathcal{C}$ lets you combine things.

- For example **Set** has $(0, +)$, $(1, \times)$, and infinitely-many others.
- Given $f\colon A \to A'$ and $g\colon B \to B'$, get

$$(f + g)\colon (A + B) \to (A' + B') \qquad (f \times g)\colon (A \times B) \to (A' \times B')$$

- So **Set** is a *distributive category*.

Monoids and comonoids

Given a cat'y $\mathcal{C}$ with a mon'l structure $(I, \odot)$, we can define (co)monoids.

- A *monoid* (m, η, μ) consists of an object $m \in \mathrm{Ob}(\mathcal{C})$,...
- ...and maps $I \xrightarrow{\eta} m$, $m \otimes m \xrightarrow{\mu} m$, satisfying unitality and associativity.
- A *comonoid* (c, ϵ, δ) consists of an object $c \in \mathrm{Ob}(\mathcal{C})$,...
- ...and maps $c \xrightarrow{\epsilon} I$, $c \xrightarrow{\delta} c \otimes c$, satisfying counitality and coassoc'ity.

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What are these for **Set**?

- A $(1, \times)$-monoid is a usual monoid; every set is uniq'ly a $(0, +)$-monoid
- The only $(0, +)$-comonoid is 0.
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Comonoids were discovered late b/c they aren't interesting in caty's like **Set**

- In **Vect** any choice of basis gives comonoid: $V \to V \otimes V$ and $V \to k$.
- We'll see that they're very interesting in other categories too.

Monads and comonads

For any category $\mathcal{C}$, the cat'y End($\mathcal{C}$) of endofunctors on $\mathcal{C}$ is monoidal.

- Objects in End($\mathcal{C}$) are functors $F\colon \mathcal{C} \to \mathcal{C}$.
- Morphisms $\alpha\colon F \Rightarrow G$ are natural transformations.
- Monoidal unit is identity $\mathrm{id}_{\mathcal{C}}$; monoidal product is composition $F \circ G$.

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A (co)monad is a (co)monoid in the monoidal category of endofunctors.

- You can see that small steps add up fast in CT.
- A monad is: a functor $F\colon \mathcal{C} \to \mathcal{C}$ and NTs $\mathrm{id}_\mathcal{C} \Rightarrow F$ and $F \circ F \Rightarrow F$.
- A comonad is: a functor $F\colon \mathcal{C} \to \mathcal{C}$ and NTs $F \Rightarrow \mathrm{id}_\mathcal{C}$ and $F \Rightarrow F \circ F$.
- Each must satisfy the respective (co)unitality and (co)assocativity.

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These come up throughout math and functional programming. Examples:

- For each alg. theory (e.g. groups), there's an associated monad on **Set**.
- Monads capture *effects* (IO, non-det'sm, exceptions) in functional PL.
- Our goal is to discuss free monads and cofree comonads.
- These'll model terminating programs and persistent machines resp.

Outline

Polynomial functors

Polynomial functors **Set** $\to$ **Set** are the closure of the identity under $\sum, \prod$.

- Let y denote $\mathrm{id}_{\mathbf{Set}}$. Then $y^A = \prod_{a \in A} y$ sends $X \mapsto X^A$, e.g. $y^0 = 1$.
- A polynomial functor is $\sum_{i \in I} \prod_{j \in J_i} y$. E.g. $y^{\mathbb{N}} + \mathbb{R}y^2 + 17$.
- We call each $i \in I$ a *position* and each $j \in J_i$ a *direction* at i.
- Maps between polynomial functors are natural transformations.
- The category **Poly** is nice because *calculation is easy*!
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Polynomials can be composed, and this is a monoidal product denoted $\circ$.

- Example: $y^2 \circ (y+1) = y^2 + 2y + 1$ and $(y+1) \circ y^2 = y^2 + 1$.
- Monoidal: given maps $p \to q$ and $p' \to q'$, get $p \circ p' \to q \circ q'$
- Example (applic'n as subst'n): $p(42) = p \circ 42y^0$.
- An (A, B)-Moore machine is a poly map $S \to By^A \circ S = B \times S^A$.

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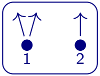
Polynomials can be tensored (Dirichlet product) $p \otimes q$, e.g. $y^3 \otimes y^3 = y^9$.

- We can use this to wire together Moore machines in block diagrams.

More on substitution product

We can draw polynomials as corolla forests. Substitution is stacking.

$p = y^2 + y$



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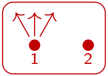
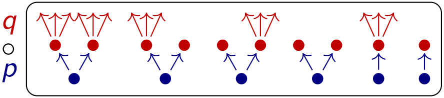


$p \circ q = y^6 + 3y^3 + 2$



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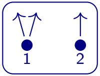

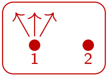

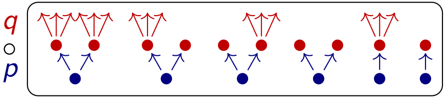

An element of $p \circ p \circ \cdots \circ p$ can be thought of as a flow-chart:

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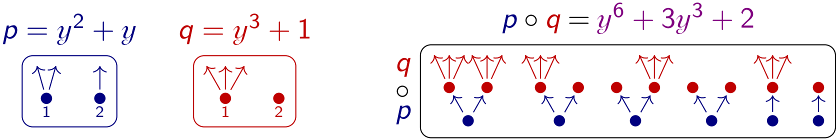

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- A comonoid includes a polynomial c and maps $c \xrightarrow{\epsilon} y$ and $c \xrightarrow{\delta} c \circ c$...
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- Idea: Ob's = positions; Mor's=directions; Id's=ϵ; Cod's & Comps=δ.

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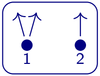

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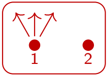

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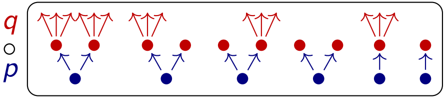

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Another is that $\circ$-monoids are (very close to) "operads".

- An *operad* $\mathcal{O}$ is a "system of operations".
- For every arity N, a set $\mathcal{O}_N$, and how they compose.

Free monad monad and cofree comonad comonad

CT tries to foreground the most general abstractions from across mathematics.

- Monad and comonad are some of the most important concepts in CT.
- Free (like free group) is ubiquitous across math; cofree is the dual notion.
- And module (e.g. vector space, group action) is also ubiquitous.
- "The free monad monad is a module over the cofree comonad comonad"
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First, let's say what the free monad and cofree comonad constructions are.

- Given a polynomial functor p, we have a monad $\mathfrak{m}_p$ and a comonad $\mathfrak{c}_p$.

$$\mathfrak{m}_p := \operatorname{colim}\left(\cdots \longleftarrow y+p\circ(y+p\circ(y+p)) \longleftarrow y+p\circ(y+p) \longleftarrow y+p \longleftarrow y\right)$$
$$\mathfrak{c}_p := \lim\left(\cdots \longrightarrow y\times p\circ(y\times p\circ(y\times p)) \longrightarrow y\times p\circ(y\times p) \longrightarrow y\times p \longrightarrow y\right)$$

- In fact $\mathfrak{m}_- : \mathbf{Poly} \to \mathbf{Poly}$ is itself a monad and $\mathfrak{c}_-$ is a comonad.
- Hence we refer to $\mathfrak{m}_-$ (resp. $\mathfrak{c}_-$) as the *(co)free (co)monad (co)monad*.

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- ...would suggest itself as a statement with theoretical significance.
- What we need to show is that it means something about working language.

First, let's say what the free monad and cofree comonad constructions are.

- Given a polynomial functor p, we have a monad $\mathfrak{m}_p$ and a comonad $\mathfrak{c}_p$.

$$\mathfrak{m}_p := \operatorname{colim}\left(\cdots \longleftarrow y+p \circ (y+p \circ (y+p)) \longleftarrow y+p \circ (y+p) \longleftarrow y+p \longleftarrow y \right)$$

$$\mathfrak{c}_p := \lim\left(\cdots \longrightarrow y \times p \circ (y \times p \circ (y \times p)) \longrightarrow y \times p \circ (y \times p) \longrightarrow y \times p \longrightarrow y \right)$$

- In fact $\mathfrak{m}_- : \mathbf{Poly} \to \mathbf{Poly}$ is itself a monad and $\mathfrak{c}_-$ is a comonad.
- Hence we refer to $\mathfrak{m}_-$ (resp. $\mathfrak{c}_-$) as the *(co)free (co)monad (co)monad*.

Theorem: There is a natural map exhibiting $\mathfrak{m}_-$ as a module over $\mathfrak{c}_-$:

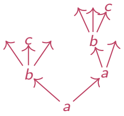
$$\mathfrak{m}_p \otimes \mathfrak{c}_q \to \mathfrak{m}_{p \otimes q}$$

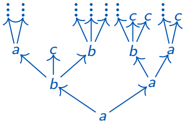
How it works

Both $\mathfrak{m}_p$ and $\mathfrak{c}_p$ are carried by poly'ls; what are their pos'ns and direc'ns?

- First let's define a *p-tree* to be a rooted tree, where each node is...
- ...labeled by a position $P \in p(1)$, and has $p[P]$-many branches.
- Each position in $\mathfrak{m}_p$ and $\mathfrak{c}_p$ can be represented by a p-tree.
  - In $\mathfrak{m}_p$, each tree is *well-founded*: always a finite path down to root
  - In $\mathfrak{c}_p$, they are generally infinite: only stops if it has no branches.

$p \coloneqq \{a\}y^2 + \{b\}y^3 + \{c\}$





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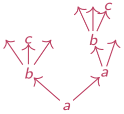

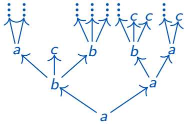

How do we think about the module structure $\mathfrak{m}_p \otimes \mathfrak{c}_q \to \mathfrak{m}_{p\otimes q}$?

- Think of $T \in \mathfrak{m}_p(1)$ as a terminating program, or a finite flowchart.
- Think of $U \in \mathfrak{c}_q(1)$ as a machine or operating system, running forever.
- We can lay T next to U and move forward through both in tandem.

We can use this to run programs that interact with a server/operating system.

- E.g. compose $\mathfrak{m}_p \otimes \mathfrak{c}_q \to \mathfrak{m}_{p\otimes q} \xrightarrow{\mathfrak{m}_\varphi} \mathfrak{m}_y \to y$ for some $p \otimes q \xrightarrow{\varphi} y$.
- This way, the program interacts with (controls) the machine.

Outline

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- What minimally-assumptive math'l foundation supports language?
- Understand the structure and dynamics of everything involved.

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Today: language as a relationship between program and machine.

- Programs terminate, machines persist; programs "run on" machines,...
- ...modeled via the (co)free (co)monad (co)monad module structure:

$$\mathfrak{m}_p \otimes \mathfrak{c}_q \to \mathfrak{m}_{p\otimes q}.$$

- CT's conciseness here suggests that this is a fundamental relationship.

Thanks! Comments and questions welcome...